Introduction to the Theory of Species of Structures

François Bergeron, Gilbert Labelle, and Pierre Leroux

November 25, 2013

UQÀM **Université du Québec à Montréal**
Département de mathématiques
Case postale 8888, Succursale Centre-Ville
Montréal (Québec) H3C 3P8

Contents

0.1 Introduction

Following a slow (and forced) evolution in the history of mathematics, the modern notion of function (due to Dirichlet, 1837) has been made independent of any actual description format. A similar process has led André Joyal [158] to introduce in combinatorics the notion of "Species" to make the description of structures independent of any specific format. The theory serves as an elegant "explanation" for the surprising power of generating function uses for the solution of structure enumeration. During the last decades considerable progress has been made in clarifying and strengthening the foundations of enumerative combinatorics. A number of useful theories, especially to explain algebraic techniques, have emerged. We mention, among others, *Möbius Inversion* (Rota [284], Rota–Smith [289], Rota–Sagan [288]), *Partitional Composition* (Cartier–Foata [45], Foata [106, 112]), *Prefabs* (Bender–Goldman [11]), *Reduced Incidence Algebras* (Mullin–Rota [253], Doubilet, Rota, and Stanley [80], Dür [84]), *Binomial Posets* and *exponential structures* (Stanley [301, 300]), *Möbius categories* (Content–Lemay–Leroux [66], Leroux [212, 214]), *Umbral Calculus* and *Hopf Algebras* (Rota [286], Joni–Rota [157]), *Pólya Theory* (Pólya [263], Redfield [275], de Bruijn [68], Robinson [282]), and *Species of Structures* (Joyal [158]). Many authors have also underlined the importance of these methods to solve problems of enumeration, in particular, Bender–Williamson [12], Berge [13], Comtet [58], Flajolet [91], Goulden–Jackson [133], Graham–Knuth–Patashnik [136], Kerber [170], Harary–Palmer [144], Knuth [172], Liu [222], Riordan [281], Moon [251], Sagan [290], Stanley [304, 302], Stanton–White [306], van Lint–Wilson [316], Wehrhahn [324], and Wilf [326].

In addition, during this same period, the subject has been greatly enriched by its interaction with algebra and theoretical computer science as a source of application and motivation. The importance of combinatorics for the analysis of algorithms and the elaboration of efficient data structures, is established in the fundamental book of Knuth [172]. A good knowledge of combinatorics is now essential to the computer scientist. Of particular importance are the following areas: formal languages, grammars and automata theory (see for instance Berstel–Reutenauer [30], Eilenberg [88], Greene [137], Lothaire [227], Reutenauer [276, 277], and the work of Schützenberger, Cori, Viennot and the Bordeaux School); asymptotic analysis and average case complexity (see Bender [9], Bender–Canfield [10]), Flajolet–Odlyzko [97], Flajolet–Salvy–Zimmermann [99], and Knuth [172], and Sedgewick–Flajolet [295]; and combinatorics of data structures (see Aho–Hopcroft–Ullman [1], Baeza-Yates–Gonnet [5], Brassard–Bratley [38], Mehlhorn [244], and Williamson [329]).

The combinatorial theory of species, introduced by Joyal in 1980, is set in this general framework. It provides a unified understanding of the use of generating series for both labeled and unlabeled structures, as well as a tool for the specification and analysis of these structures. Of particular importance is its capacity to transform recursive definitions of (tree-like) structures into functional or differential equations, and conversely. Encompassing the description of structures together with permutation group actions, the theory of species conciliates the calculus of generating series and functional equations with Pólya theory, following previous efforts to establish an algebra of cycle index series, particularly by de Bruijn [68] and Robinson [282]. This is achieved by extending the concept of group actions to that of functors defined on groupoids, in this case the category of finite sets and bijections. The functorial concept of species of structures goes back

to Ehresmann [87]. The functorial property of combinatorial constructions on sets is also pointed out in a paper of Mullin and Rota [253] in the case of reluctant functions, a crucial concept for the combinatorial understanding of Lagrange inversion. There are also links between the algebra of operations on species and category theory. For example, the partitional composition of species can be described in the general settings of doctrines (see Kelly [166]), operads (see May [233] and Loday [224]), and analytic functors (see Joyal [163]).

Informally, a species of structures is a rule, F, associating with each finite set U, a finite set $F[U]$ which is "independent of the nature" of the elements of U. The members of the set $F[U]$, called F-*structures*, are interpreted as combinatorial structures on the set U given by the rule F. The fact that the rule is independent of the nature of the elements of U is expressed by an invariance under relabeling. More precisely, to any bijection $\sigma : U \longrightarrow V$, the rule F associates a bijection $F[\sigma] : F[U] \longrightarrow F[V]$ which transforms each F-structure on U into an (isomorphic) F-structure on V. It is also required that the association $\sigma \mapsto F[\sigma]$ be consistent with composition of bijections. In this way the concept of species of structures puts as much emphasis on isomorphisms as on the structures themselves. In categorical terms, a species of structures is simply a functor from the category $\mathbb{B}$ of finite sets and bijections to itself.

As an example, the class $\mathcal{G}$ of simple (finite) graphs and their isomorphisms, in the usual sense, gives rise to the species of graphs, also denoted $\mathcal{G}$. For each set U, the elements of $\mathcal{G}[U]$ are just the simple graphs with vertex set U. For each $\sigma : U \longrightarrow V$, the bijection $\mathcal{G}[\sigma] : \mathcal{G}[U] \longrightarrow \mathcal{G}[V]$ transforms each simple graph on U into a graph on V by relabeling via σ. Similarly, any class of discrete structures closed under isomorphisms gives rise to a species.

Furthermore, species of structures can be combined to form new species by using set theoretical constructions. There results a variety of combinatorial operations on species, including addition, multiplication, substitution, derivation, etc, which extend the familiar calculus of formal power series. Indeed to each species of structures, we can associate various formal power series designed to treat enumeration problems of a specific kind (labeled, unlabeled, asymmetric, weighted, etc.). Of key importance is the fact that these associated series are "compatible" with operations on species. Hence each (algebraic, functional or differential) identity between species implies identities between their associated series. This is in the spirit of Euler's method of generating series.

For example, let $\mathfrak{a}$ denote the species of *trees* (acyclic connected simple graphs) and $\mathcal{A}$, that of *rooted trees* (trees with a distinguished vertex). Then the functional equation

$$\mathcal{A} = X\,E(\mathcal{A}), \tag{1}$$

expresses the basic fact that any rooted tree on a finite set U can be naturally described as a root (a vertex $x \in U$) to which is attached a set of disjoint rooted trees (on $U \setminus \{x\}$), see Figure 2.9. Equation (1) yields immediately the following equalities between generating series

$$\mathcal{A}(x) = x\,e^{\mathcal{A}(x)}, \qquad T(x) = x\exp(\sum_{k \geq 0} \frac{T(x^k)}{k}).$$

These formulas go back to Cayley [46] and Pólya [263]. The first refers to the exponential generating

series $\mathcal{A}(x) = \sum_{n\geq 0} a_n\, x^n/n!$, where a_n is the number of rooted trees on a set of n elements (labeled rooted trees), and yields Cayley's formula $a_n = n^{n-1}$ via the Lagrange inversion formula. The second refers to the ordinary generating series $T(x) = \sum_{n\geq 0} T_n\, x^n$, where T_n is the number of isomorphism types of rooted trees (unlabeled rooted trees) on n elements, and yields a recurrence formula for these numbers.

Analogously, the identity

$$2(n-1)n^{n-2} = \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1},$$

and Otter's formula [259]

$$t(x) = T(x) + \frac{1}{2}(T(x^2) - T^2(x)),$$

where $t(x) = \sum_{n\geq 1} t_n\, x^n$ is the ordinary generating series of the number t_n of unlabeled trees on n elements, both follow from the species isomorphism

$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

allowing us to express the species $\mathfrak{a}$ of trees as a function of the species of rooted trees. We call this identity the *dissymmetry* theorem for trees (see Leroux [213], Leroux and Miloudi [215]). It is inspired from the dissimilarity formula of Otter [259] and the work of Norman [257] and Robinson [282] on the decomposition of graphs into 2-connected components.

Since its introduction, the theory of species of structures has been the focus of considerable research by the Montréal school of combinatorics as well as numerous other researchers. The goal of this text is to present the basic elements of the theory, and to give a unified account of some of it's developments and applications.

Chapter 1 contains the first key ideas of the theory. A general discussion on the notion of discrete structures leads naturally to the formal definition of species of structures. Some of the basic formal power series associated to a species F are introduced: the (exponential) generating series $F(x)$ for labeled enumeration, the type generating series $\widetilde{F}(x)$ for unlabeled enumeration, and the cycle index series $Z_F(x_1, x_2, x_3, \ldots)$ as a general enumeration tool. Finally, we introduce the combinatorial operations of addition, multiplication, substitution (partitional composition) and derivation of species of structures. These operations extend and interpret in the combinatorial context of species the corresponding operations on formal power series.

Chapter 2 begins with an introduction to three other operations: pointing, cartesian product and functorial composition. Pointing is a combinatorial analogue of the operator $x(d/dx)$ on series. The cartesian product, consisting of superposition of structures, corresponds to the *Hadamard product* of series (coefficient-wise multiplication). Functorial composition, not to be confused with substitution, is the natural composition of species considered as functors (see Décoste–Labelle–Leroux [74]). Many species of graphs and multigraphs can be expressed easily by this operation.

The theory is then extended in Chapter 3 to weighted species where structures are counted according to certain parameters, and to multisort species in analogy with functions of several variables. These generalizations broaden the range of applications to more refined enumeration problems.

For example, the generating series for Laguerre polynomials,

$$\sum_{n\geq 0}\mathcal{L}_n^{(\alpha)}(t)\frac{x^n}{n!} = \left(\frac{1}{1-x}\right)^{\alpha+1}\exp\left(-t\,\frac{x}{1-x}\right).$$

suggests a combinatorial "model" for these polynomials, consisting of permutations (with cycle counter $\alpha + 1$) and oriented chains (each with weight $-t$). This model gives rise to a combinatorial theory of Laguerre polynomials, where identities appear as consequences of elementary constructions on discrete structures. The same approach can be applied to many other families of polynomials. See, for example, Bergeron [20], Dumont [83], Foata [107, 109], Foata–Labelle [110], Foata–Leroux [111], Foata–Schützenberger [112], Foata–Strehl [113, 114], Foata–Zeilberger [115], Labelle–Yeh [202, 204, 208], Leroux–Strehl [216], Strehl [308, 309, 310], Viennot [319], and Zeng [340]. Following those lines, the tex contains a combinatorial treatment of Eulerian, Hermite, Laguerre, and Jacobi polynomials.

Chapter 1

Introduction to Species of Structures

This chapter contains the basic concepts of the combinatorial theory of species of structures. It is an indispensable starting point for the developments and applications presented in the subsequent chapters. We begin with some general considerations on the notion of structure, everywhere present in mathematics and theoretical computer science. These preliminary considerations lead us in a natural manner to the fundamental concept of species of structures. The definition of species puts the emphasis on the transport of structures along bijections and is due to C. Ehresmann [87], but it is A. Joyal [158] who showed its effectiveness in the combinatorial treatment of formal power series and for the enumeration of labelled structures as well as unlabeled (isomorphism types of) structures.

We introduce in Section 1.2 some of the first power series that can be associated to species: generating series, types generating series, cycle index series. They serve to encode all the information concerning labelled and unlabeled enumeration.

1.1 Species of Structures

The concept of structure is fundamental, recurring in all branches of mathematics, as well as in computer science. From an informal point of view, a **structure** s is a construction γ which one performs on a set U (of data). It consists of a pair $s = (\gamma, U)$. It is customary to say that U is the underlying set of the structure s or even that s is a structure constructed from (or labelled by) the set U. Figure 1.1 depicts two examples of structures: a rooted tree and an oriented cycle. In a set theoretical fashion, the tree in question can be described as $s = (\gamma, U)$, where

$$
\begin{aligned}
U &= \{a, b, c, d, e, f\}, \\
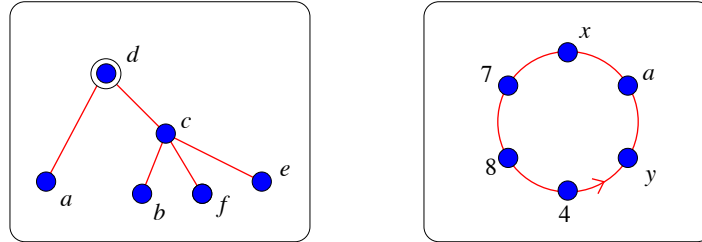\gamma &= (\{d\}, \{\{d, a\}, \{d, c\}, \{c, b\}, \{c, f\}, \{c, e\}\}).
\end{aligned}
$$

Figure 1.1: Two examples of combinatorial structures.

The singleton $\{d\}$ which appears as the first component of γ indicates the **root** of this rooted tree. As for the oriented cycle, it can be put in the form $s = (\gamma, U)$, where

$$U = \{x, 4, y, a, 7, 8\},$$
$$\gamma = \{(4, y), (y, a), (a, x), (x, 7), (7, 8), (8, 4)\}.$$

The abuse of notation $s = \gamma$, which consists of identifying a structure $s = (\gamma, U)$ with the construction γ, will be used if it does not cause any ambiguity with regard to the nature of the underlying set U. As an example which could give rise to such an ambiguity, consider the structure

$$s = (\gamma, U) \quad \text{with} \quad U = \{c, x, g, h, m, p, q\}, \text{ and } \gamma = \{x, m, p\},$$

so that γ is a subset of U. Clearly the knowledge of γ by itself does not enable one to recover the underlying overset U. A traditional approach to the concept of structure consists in generalizing the preceding examples within axiomatic set theory. However, in the present work we adopt a *functorial* approach which puts an emphasis on the *transport of structures* along bijections. Here is an example which illustrates the concept of transport of structures.

Example 1.1. Consider the rooted tree $s = (\gamma, U)$ of Figure 1.1 whose underlying set is $U = \{a, b, c, d, e, f\}$. Replace each element of U by those of $V = \{x, 3, u, v, 5, 4\}$ via the bijection $\sigma : U \longrightarrow V$ described by Figure 1.1. This figure clearly shows how the bijection σ allows the transport of the rooted tree s onto a corresponding rooted tree $t = (\tau, V)$ on the set V, simply by replacing each vertex $u \in U$ by the corresponding vertex $\sigma(u) \in V$. We say that the rooted tree t has been obtained by transporting the rooted tree s along the bijection σ, and we write $t = \sigma \cdot s$. From a purely set theoretical point of view, this amounts to replacing simultaneously each element u of U appearing in γ by the corresponding element $\sigma(u)$ of V in the expression of γ. The rooted trees s and t are said to be isomorphic, and σ is said to be an isomorphism between s and t.

Intuitively two isomorphic structures can be considered as identical if the nature of the elements of their underlying sets is ignored. This "general form" that isomorphic structures have in common is their isomorphism type. It often can be represented by a diagram (see, for example, Figure 1.3)) in which the elements of the underlying set become "indistinguishable" points. The structure is then said to be unlabeled. Figure 1.4 illustrates a rooted tree automorphism. In this case, the sets
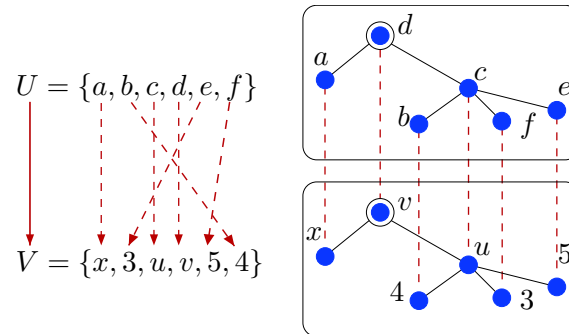
Figure 1.2: Transport as a relabeling of vertices.
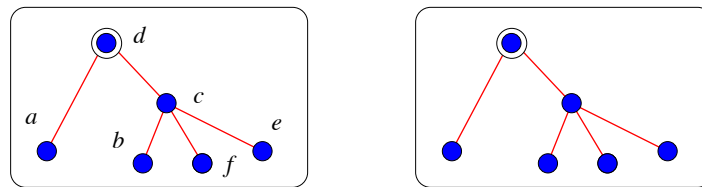
Figure 1.3: Labelled and unlabeled structures.

U and V coincide, the bijection $\sigma : U \longrightarrow U$ is a permutation of U, and the transported rooted tree $\sigma \cdot s$ is identical to the tree s, that is to say, $s = \sigma \cdot s$. The preceding examples show that the concept of transport of structures is of prime importance since it enables one to define the notions of isomorphism, isomorphism type and automorphism. In fact, the transport of structures is at the very base of the general concept of species of structures.

Figure 1.4: A non trivial tree automorphism.

Example 1.2. As an introduction to the formal definition of species of structures, here is a detailed description of the *species* $\mathcal{G}$ of all **simple graphs** (i.e., undirected graphs without loops or multiple edges). For each finite set U, we denote by $\mathcal{G}[U]$ the set of all structures of simple graph on U.

Thus

$$\mathcal{G}[U] = \{g \mid g = (\gamma, U),\ \gamma \subseteq \wp^{[2]}[U]\},$$
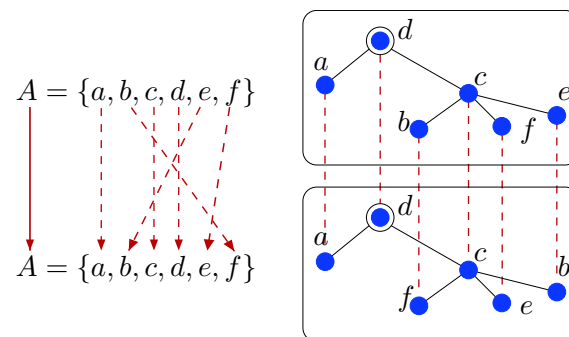
where $\wp^{[2]}[U]$ stands for the collection of (unordered) pairs of elements of U. In the simple graph $g = (\gamma, U)$, the elements of U are the **vertices** of the graph g, and γ is the set of its **edges**. Clearly $\mathcal{G}[U]$ is a finite set. The following three expressions are considered to be equivalent:

- g is a simple graph on U;
- $g \in \mathcal{G}[U]$;
- g is a $\mathcal{G}$-structure on U.

Moreover, each bijection $\sigma : U \longrightarrow V$ induces, by transport of structure (see Figure 1.5), a function

$$\mathcal{G}[\sigma] : \mathcal{G}[U] \longrightarrow \mathcal{G}[V], \qquad g \mapsto \sigma \cdot g,$$

describing the transport of graphs along σ. Formally, if $g = (\gamma, U) \in \mathcal{G}[U]$, then $\mathcal{G}[\sigma](g) = \sigma \cdot g = (\sigma \cdot \gamma, V)$, where $\sigma \cdot \gamma$ is the set of pairs $\{\sigma(x), \sigma(y)\}$ of elements of V obtained from pairs $\{x, y\} \in \gamma$. Thus each edge $\{x, y\}$ of g finds itself relabeled $\{\sigma(x), \sigma(y)\}$ in $\sigma \cdot g$. Since this transport of graphs along σ is only a relabeling of the vertices and edges by σ, it is clear that for bijections $\sigma : U \longrightarrow V$ and $\tau : V \longrightarrow W$, one has

$$\mathcal{G}[\tau \circ \sigma] = \mathcal{G}[\tau] \circ \mathcal{G}[\sigma], \tag{1.1}$$

and that, for the identity map $\mathrm{Id}_U : U \longrightarrow U$, one has

$$\mathcal{G}[\mathrm{Id}_U] = \mathrm{Id}_{\mathcal{G}[U]}. \tag{1.2}$$

These two equalities express the functoriality of the transports of structures $\mathcal{G}[\sigma]$. It is this property which is abstracted in the definition of species of structures.
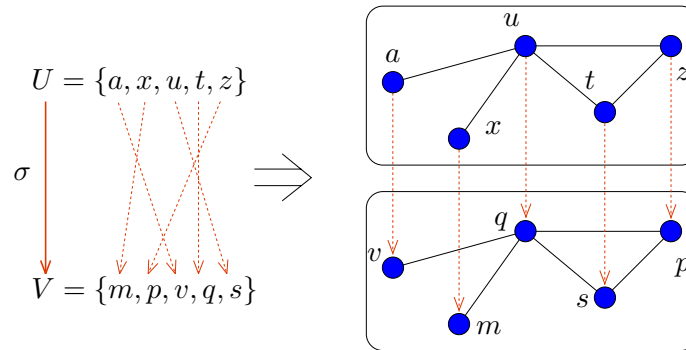

Figure 1.5: Relabeling of vertices.

1.1.1 General definition of species of structures

Definition 1.3. A **species of structures** is a rule F which produces

 i) for each finite set U, a finite set $F[U]$,

 ii) for each bijection $\sigma : U \longrightarrow V$, a function $F[\sigma] : F[U] \longrightarrow F[V]$.

The functions $F[\sigma]$ should further satisfy the following functorial properties:

 a) for all bijections $\sigma : U \longrightarrow V$ and $\tau : V \longrightarrow W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma], \tag{1.3}$$

 b) for the identity map $\mathrm{Id}_U : U \longrightarrow U$,

$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}. \tag{1.4}$$

An element $s \in F[U]$ is called an F-**structure** on U (or even a **structure of species** F on U). The function $F[\sigma]$ is called the **transport** of F-structures along σ. The advantage of this definition of species is that the rule F, which produces the structures $F[U]$ and the transport functions $F[\sigma]$, can be described in any fashion provided that the functoriality conditions (1.3) and (1.4) hold. For example, one can either use axiomatic systems, explicit constructions, algorithms, combinatorial operations, functional equations or even simple geometric figures to specify a species. We will illustrate below each of these approaches with some examples. It immediately follows from its functorial properties that each transport function $F[\sigma]$ is necessarily a bijection (See Exercise 1.2). We use the notation $\sigma{\cdot}s$, or sometimes $\sigma{\cdot}_F s$ to avoid ambiguity, to designate $F[\sigma](s)$. The following statements are to be considered equivalent:

 – s is a structure of species F on U;

 – $s \in F[U]$;

 – s is an F-structure on U.

In order to represent a generic F-structure, we often utilize drawings like those of Figures 1.6 The black dots in these figures represent the (distinct) elements of the underlying set. The F-structure itself is represented in Figure 1.6 a) by a circular arc labelled F, and in Figure 1.6 b) by the superposition of the symbol F. Observe that the notions of isomorphism, isomorphism type and automorphism of F-structures are implicitly contained in the definition of the species F.

Definition 1.4. Consider two F-structures $s_1 \in F[U]$ and $s_2 \in F[V]$. A bijection $\sigma : U \longrightarrow V$ is called an **isomorphism** of s_1 to s_2 if $s_2 = \sigma{\cdot}s_1 = F[\sigma](s_1)$. One says that these structures have the same *isomorphism type*. Moreover, an isomorphism from s to s is said to be an **automorphism** of s.
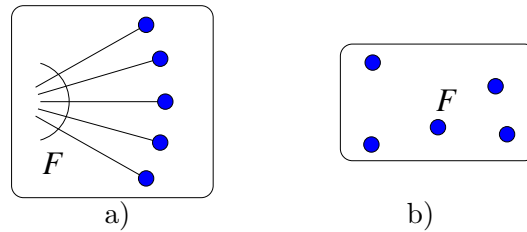
Figure 1.6: Possible representations of typical F-structures.

1.1.2   Species described through set theoretic axioms

A species F can be defined by means of a system $\mathfrak{A}$ of well-chosen axioms by requiring

$$s = (\gamma, U) \in F[U] \quad \text{if and only if} \quad s = (\gamma, U) \quad \text{is a model of} \quad \mathfrak{A}.$$

The transport $F[\sigma]$ is carried out in the natural fashion illustrated earlier. Clearly one can introduce in this manner a multitude of species, including the following (see Exercise 1.4):

- the species $\mathcal{A}$, of **rooted trees**;
- the species $\mathcal{G}$, of **simple graphs**;
- the species $\mathcal{G}^c$, of **connected simple graphs**;
- the species $\mathfrak{a}$, of **trees** (connected simple graphs without cycles);
- the species $\mathcal{D}$, of **directed graphs**;
- the species Par, of **set partitions**;
- the species $\wp$, of **subsets**, i.e., $\wp[U] = \{S \mid S \subseteq U\}$;
- the species End, of **endofunctions**, i.e., $\text{End}[U] = \{\psi \mid \psi : U \longrightarrow U\}$;
- the species Inv, of **involutions**, i.e., those endofunctions ψ such that $\psi \circ \psi = Id$;
- the species $\mathcal{S}$, of **permutations** (i.e., bijective endofunctions);
- the species $\mathcal{C}$, of **cyclic permutations** (or oriented cycles);
- the species L, of **linear** (or **total orders**).

For example, we can describe the set of endofunctions on U by the set theoretic characterization: $(\psi, U) \in \text{End}[U]$ if and only if

$$\psi \subseteq U \times U \quad \text{and} \quad (\forall x)[(x \in U) \implies (\exists! y)[(y \in U) \quad \text{and} \quad ((x, y) \in \psi)]]. \tag{1.5}$$

Directed graphs ψ satisfying (1.5) are called **functional digraphs**. We also say that ψ is thesagittal graph of the endofunction. Note that the transport $\text{End}[\sigma] : \text{End}[U] \longrightarrow \text{End}[V]$ along the bijection $\sigma : U \longrightarrow V$ is given by the formula $\text{End}[\sigma](\psi) = \sigma \circ \psi \circ \sigma^{-1}$, for each $\psi \in \text{End}[U]$. Indeed, upon

setting $\theta = \mathrm{End}[\sigma](\psi) \in \mathrm{End}[V]$, the pairs $(u, \psi(u))$ run over the functional digraphs determined by ψ if and only if the pairs $(\sigma(u), \sigma(\psi(u))$ run over the sagittal graphs determined by θ. Moreover, the relation $v = \sigma(u)$ is equivalent to the relation $u = \sigma^{-1}(v)$. We then deduce that the functional digraph of θ is given by pairs of the form $(v, \sigma \circ \psi \circ \sigma^{-1}(v))$ with $v \in V$.

1.1.3   Explicit constructions of species

When the structures of a species F are particularly simple or not numerous, it can be advantageous to define the species by an explicit description of the sets $F[U]$ and transport functions $F[\sigma]$. The following species fall under this category. In each case the transport of structures $F[\sigma]$ is obvious.

- The species E, of **sets**, is simply defined as $E[U] := \{U\}$. Thus, for each finite set U, there is a unique E-structure, namely the set U itself.

- The species ε, of **elements**, is defined as $\varepsilon[U] := U$. Hence the associated structures on U are simply the elements of U.

- The species X, characteristic of **singletons**, is defined by setting

$$X[U] := \begin{cases} \{U\}, & \text{if } \#U = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

  In other words there are no X-structure on sets having cardinality other then 1.

- The species 1, characteristic of the empty set, defined by

$$1[U] := \begin{cases} \{U\}, & \text{if } U = \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

- The **empty**species!empty species, denoted by 0, simply defined as $0[U] := \emptyset$ for all U.

- The species E_2, characteristic of sets of cardinality 2, defined by

$$E_2[U] := \begin{cases} \{U\}, & \text{if } \#U = 2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

1.1.4   Algorithmic descriptions

One can specify structures in an algorithmic fashion. For instance, an algorithm can be given which generates all the binary rooted trees on a given set of vertices. If we designate this algorithm by $\mathcal{B}$, then for an input set U, the output $\mathcal{B}[U]$ is the set of $\mathcal{B}$-structures, namely all possible binary graphs on U. For each bijection $\sigma : U \longrightarrow V$, algorithm $\mathcal{B}$ should also produce an explicit and

effective translation of each element of $\mathcal{B}[U]$ into an element of $\mathcal{B}[V]$. For example, one of the structures produced by algorithm $\mathcal{B}$ on the set $U = \{a, b, c, d, e, f\}$ would be

$$((\emptyset, b, (\emptyset, d, \emptyset)), c, (\emptyset, a, ((\emptyset, f, \emptyset), e, \emptyset))).$$

(see Figure 1.7 for a representation of this binary rooted tree). The transport of this structure along the bijection $\sigma : \{a, b, c, d, e, f\} \longrightarrow \{A, B, C, D, E, F\}$, simply replaces each letter by its corresponding capital letter:

$$\mathcal{B}[\sigma]((\emptyset, b, (\emptyset, d, \emptyset)), c, (\emptyset, a, ((\emptyset, f, \emptyset), e, \emptyset))) = ((\emptyset, B, (\emptyset, D, \emptyset)), C, (\emptyset, A, ((\emptyset, F, \emptyset), E, \emptyset))).$$

In general, species of structures satisfying a "functional equation" can readily be defined in an
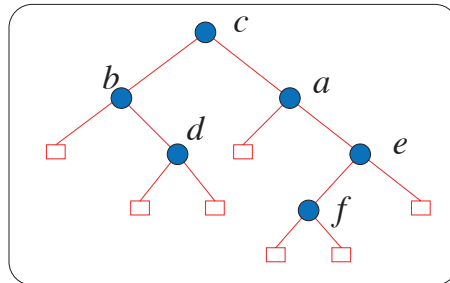
Figure 1.7: A binary rooted tree.

algorithmic or recursive manner.

1.1.5   Using combinatorial operations on species

Another way of producing species of structures is by applying operations to known species. These operations (addition, multiplication, substitution, differentiation, etc.) will be described in detail in Sections 2.1, 2.2, 2.3, and 2.4. Here are some examples:

  – the species E^3, of **tricolorations**, $E^3 = E \cdot E \cdot E$,
  – the species E_+, of **non-empty sets**, $1 + E_+ = E$,
  – the species $\mathcal{H}$, of **hedges** (or **lists of rooted trees**), $\mathcal{H} = L(\mathcal{A})$,
  – the species Der, of **derangements**, $E \cdot \text{Der} = \mathcal{S}$,
  – the species Bal, of **ballots** (or **ordered partitions**).

1.1.6   Functional equation solutions

It also frequently happens that species of structures are described or characterized recursively by functional equations. Here are some examples:

- the species $\mathcal{A}$, of rooted trees, $\mathcal{A} = X \cdot E(\mathcal{A})$,
- the species L, oflinear orders, $L = 1 + X \cdot L$,
- the species $\mathcal{A}_L$, of **ordered rooted trees**, $\mathcal{A}_L = X \cdot L(\mathcal{A}_L)$,
- the species $\mathcal{B}$, of **binary rooted trees**, $\mathcal{B} = 1 + X \cdot \mathcal{B}^2$,
- the species $\mathcal{P}$, of **commutative parenthesizations**, $\mathcal{P} = X + E_2(\mathcal{P})$.

The description of species of structures with the help of functional equations plays a central role in the theory of species.

1.1.7 Geometric descriptions

One can sometimes gain in simplicity or clarity by describing a species F with the help of one (or several) figure(s) which schematically represents a typical F-structure. Figure 1.8 represents a typical structure belonging to the species P of **polygons** (i.e., non-oriented cycles) on a set of cardinality 5. By definition, $P[U]$ is the set of polygons on U.
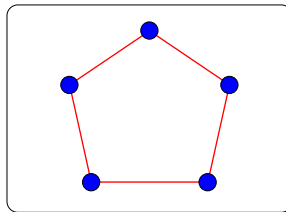
Figure 1.8: A polygon.

Remark 1.5. The reader who is familiar with category theory will have observed that a species is simply a functor $F : \mathbb{B} \longrightarrow \mathbb{E}$ from the category $\mathbb{B}$ of finite sets and bijections to the category $\mathbb{E}$ of finite sets and functions. Although the knowledge of category theory is not necessary in order to read this book, the interested reader is encouraged to consult a basic text on category theory, such as Maclane [229].

1.2 Associated Series

We will now associate to each species of structures F three important formal power series related to the enumeration of F-structures. An F-structure $s \in F[U]$ on a set U is often referred to as a **labelled** structure, whereas an **unlabeled** structure is an isomorphism class of F-structures. The three series are

- the (exponential) generating series of F, denoted $F(x)$, for labelled enumeration,

    – the type generating series of F, denoted $\widetilde{F}(x)$, for unlabeled enumeration,

    – the cycle index series of F, denoted $Z_F(x_1, x_2, x_3, \ldots)$, as a general enumeration tool.

These series serve to "encode" all the information concerning the enumeration of labelled or unlabeled F-structures (i.e., up to isomorphism). Note that for all finite sets U, the number of F-structures on U only depends upon the number of elements of U (and not on the elements of U). In other words,

the cardinality of $F[U]$ only depends upon that of U.

This property immediately follows from the earlier observation (See Exercise 1.12) that transport functions $F[\sigma]$ are always bijections. Hence, the cardinalities $|F[U]|$ are completely characterized by the sequence of values $f_n = |F[\{1, 2, \ldots, n\}]|$, $n \geq 0$. For ease in notation, let us write $[n]$ to designate the set $\{1, 2, 3, \ldots, n\}$, and $F[n]$ to designate the set $F[\{1, 2, \ldots, n\}]$, rather than $F[[n]]$.

1.2.1 Generating series of a species of structures

Definition 1.6. The **generating series** of a species of structures F is the formal power series

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}, \tag{1.6}$$

where $f_n = |F[n]| = $ the number of F-structures on a set of n elements (labelled structures). Note that this series is of exponential type in the indeterminate x in the sense that $n!$ appears in the denominator of the term of degree n. The series $F(x)$ is also called the **exponential generating series** of the species F. The following notation is used to designate the coefficients of formal power series. For an ordinary formal power series

$$G(x) = \sum_{n \geq 0} g_n \, x^n,$$

we set

$$[x^n] \, G(x) = g_n. \tag{1.7}$$

For a formal power series of exponential type, of the form (1.6), we then have

$$n! \, [x^n] \, F(x) = f_n. \tag{1.8}$$

Taking the Taylor expansion (at the origin) of $F(x)$ shows that

$$n! \, [x^n] \, F(x) = \left. \frac{d^n F(x)}{dx^n} \right|_{x=0}.$$

More generally, for a formal power series in any number of variables expressed in the form

$$H(x_1, x_2, x_3, \ldots) = \sum_{n_1, n_2, n_3, \ldots} h_{n_1, n_2, n_3, \ldots} \frac{x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots}{c_{n_1, n_2, n_3, \ldots}},$$

where $c_{n_1, n_2, n_3, \ldots}$ is a given family of non-zero scalars, we have

$$c_{n_1, n_2, n_3, \ldots} \left[x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots \right] H(x_1, x_2, x_3, \ldots) = h_{n_1, n_2, n_3, \ldots}. \tag{1.9}$$

Example 1.7. Referring to species described earlier, it is easy to verify by direct enumeration the following identities:

$$
\begin{aligned}
&\text{a) } L(x) = \frac{1}{1-x}, &&\text{b) } \mathcal{S}(x) = \frac{1}{1-x}, &&\text{c) } \mathcal{C}(x) = -\log(1-x),\\
&\text{d) } E(x) = e^x, &&\text{e) } \varepsilon(x) = xe^x, &&\text{f) } \wp(x) = e^{2x},\\
&\text{g) } X(x) = x, &&\text{h) } 1(x) = 1, &&\text{i) } 0(x) = 0,\\
&\text{j) } \mathcal{G}(x) = \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}, &&\text{k) } \mathcal{D}(x) = \sum_{n \geq 0} 2^{n^2} \frac{x^n}{n!}, &&\text{l) } \mathrm{End}(x) = \sum_{n \geq 0} n^n \frac{x^n}{n!}.
\end{aligned}
\tag{1.10}
$$

The computation of the generating series for other species $\mathcal{G}^c$, Par, Inv, $\mathcal{A}$, etc. which have been mentioned earlier is less direct. It will be done after the introduction of combinatorial operations on species of structures.

1.2.2 Type generating series

Let us now consider the enumeration of isomorphism types of F-structures. We may restrict ourselves to structures on sets of the form $U = \{1, 2, \ldots, n\} = [n]$. One defines an equivalence relation $\sim$ on the set $F[n]$ by setting, for $s, t \in F[n]$,

$s \sim t$ if and only if s and t have the same isomorphism type.

In other words (see Definition 1.4), $s \sim t$ if and only if there exists a permutation $\pi : [n] \longrightarrow [n]$ such that $F[\pi](s) = t$. By definition, an **isomorphism type** of F-structures of order n is an equivalence class (modulo the relation $\sim$) of F-structures on $[n]$. Such an equivalence class is also called an unlabeled F-structure of order n. Denote by $\mathrm{T}(F_n)$ the quotient set $F[n]/\sim$, of types of F-structures of order n and let

$$\mathrm{T}(F) = \sum_{n \geq 0} \mathrm{T}(F_n).$$

Definition 1.8. The **(isomorphism) type generating series** of a species of structures F is the formal power series

$$\widetilde{F}(x) = \sum_{n \geq 0} \widetilde{f_n} x^n,$$

where $\widetilde{f_n} = |\mathrm{T}(F_n)|$ is the number unlabeled F-structures of order n. The notation $F^{\sim}(x)$ will sometimes be used for typographical reasons. Note that this is an ordinary formal power series (i.e., without factorials in the denominators) in one indeterminate x, which acts as a point counter.

Example 1.9. Direct calculations yield the following type generating series (see Exercise 1.13):

$$
\begin{aligned}
&\text{a) } \widetilde{L}(x) = \frac{1}{1-x}, \quad \text{b) } \widetilde{S}(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}, \quad \text{c) } \widetilde{C}(x) = \frac{x}{1-x}, \\
&\text{d) } \widetilde{E}(x) = \frac{1}{1-x}, \quad \text{e) } \widetilde{\in}(x) = \frac{x}{1-x}, \qquad \text{f) } \widetilde{\wp}(x) = \frac{1}{(1-x)^2}, \\
&\text{g) } \widetilde{X}(x) = x, \qquad \text{h) } \widetilde{1}(x) = 1, \qquad\qquad \text{i) } \widetilde{0}(x) = 0.
\end{aligned}
\tag{1.11}
$$

Despite the fact that the generating series of the species L and S coincide, $L(x) = S(x) = 1/(1-x)$, equality does not hold for the type generating series:

$$\widetilde{L}(x) \neq \widetilde{S}(x).$$

This provides evidence that the species L and S are not the same. Indeed, total orders and permutations are not transported in the same manner along bijections. In particular, a total order only admits a single automorphism, whereas in general a permutation admits many automorphisms. Thus there is an essential difference between permutations π of a set U of cardinality n and lists without repetition $\pi_1 \pi_2 \ldots \pi_n$ of the elements of U. Of course, if the set U happens to be given a fixed order, one can establish a bijection (depending on this order) between permutations and lists (see Example 1.18).

1.2.3 Cycle index series

In general, explicit or recursive calculation of type generating series is difficult. It requires the use of combinatorial operations on species of structures and of a third kind of series associated with each species F, the cycle index series of F, denoted by Z_F. This is a formal power series in an infinite number of variables $x_1, x_2, x_3, \ldots$. It contains more information than both series $F(x)$ and $\widetilde{F}(x)$. We first define the cycle type of a permutation.

Definition 1.10. Let U be a finite set and σ, a permutation of U. The **cycle type** of the permutation σ is the sequence $(\sigma_1, \sigma_2, \sigma_3, \ldots)$, where for $k \geq 1$, $\sigma_k = $ is the number of cycles of length k in the decomposition of σ into disjoint cycles. Observe that σ_1 is the number of fixed points of σ. Moreover, if $|U| = n$ then $\sigma_k = 0$ if $k \geq n$. The cycle type of σ can then be written in the form of a vector with n components, $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. We use the following notation:

$$
\begin{aligned}
\mathrm{Fix}\ \sigma &= \{u \in U \mid \sigma(u) = u\}, \\
\mathrm{fix}\ \sigma &= |\mathrm{Fix}\ \sigma|.
\end{aligned}
$$

$\mathrm{Fix}\ \sigma$ denotes the set of fixed points of σ, whereas $\mathrm{fix}\ \sigma = \sigma_1$ denotes the number of fixed points of σ. Figure 1.9 shows a permutation of type $(3, 4, 0, 3, 2)$.
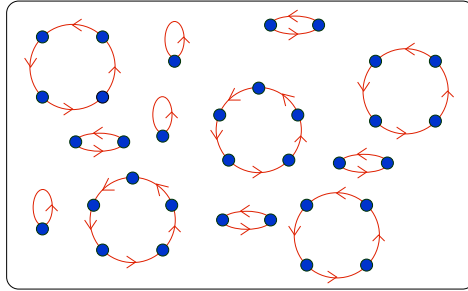
Figure 1.9: A permutation of cycle type $(3, 4, 0, 3, 2)$.

Now let F be any species. Each permutation σ of U induces, by transport of structures, a permutation $F[\sigma]$ of the set $F[U]$ of F-structures on U. Consider, for illustrative purposes, the species Inv of involutions (i.e., the endofunctions ψ such that $\psi \circ \psi = Id$) and the permutation σ of the set $U = \{a, b, c, d, e\}$ given by Figure 1.10. This permutation σ induces a permutation $\text{Inv}[\sigma]$
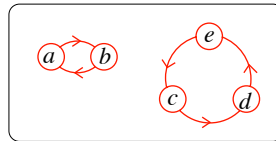

Figure 1.10: A permutation of the set $\{a, b, c, d, e\}$.

on $\text{Inv}[U]$ given by Figure 1.11 in which involutions are represented by simple graphs with each vertex having degree ≤ 1. The permutation σ is of type $(0, 1, 1)$ and permutes 5 points, while the permutation $\text{Inv}[\sigma]$ is of type $(2, 0, 2, 0, 0, 3)$ permuting the 26 involutions of U. In particular, we have $\text{fix Inv}[\sigma] = 2$.

Definition 1.11. The **cycle index series** of a species of structures F is the formal power series (in an infinite number of variables $x_1, x_2, x_3, \ldots$)

$$Z_F(x_1, x_2, x_3, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathcal{S}_n} \text{fix } F[\sigma] \, x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \ldots \right), \tag{1.12}$$

where $\mathcal{S}_n$ denotes the group of permutations of $[n]$ (i.e., $\mathcal{S}_n = \mathcal{S}[n]$) and $\text{fix } F[\sigma] = (F[\sigma])_1 = $ is the number of F-structures on $[n]$ fixed by $F[\sigma]$, i.e., the number of F-structures on $[n]$ for which σ is an automorphism.

Example 1.12. Without the help of various techniques developed in the following sections, direct calculation of cycle index series can only be carried out in very simple cases. For instance, for the
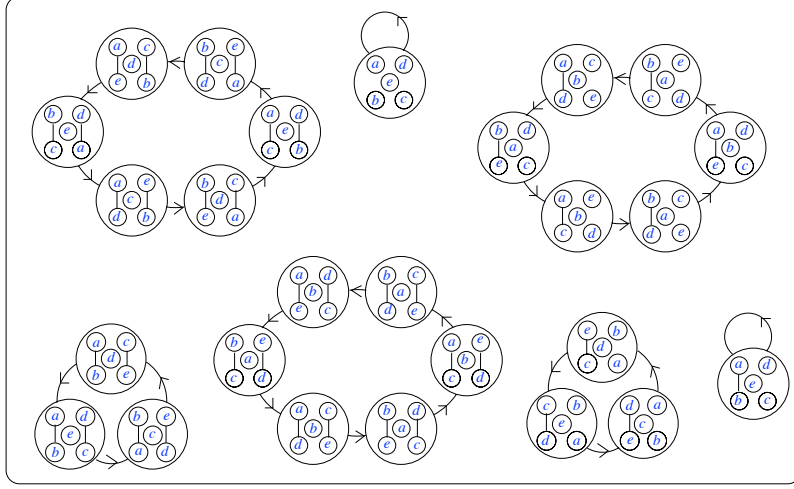
Figure 1.11: Action of $\text{Inv}[\sigma]$ on $\text{Inv}[U]$.

species 0, 1, X, L, $\mathcal{S}$, E, $\in$, we have

$$
\begin{aligned}
&\text{a) } Z_0(x_1, x_2, x_3, \ldots) = 0, \\
&\text{b) } Z_1(x_1, x_2, x_3, \ldots) = 1, \\
&\text{c) } Z_X(x_1, x_2, x_3, \ldots) = x_1, \\
&\text{d) } Z_L(x_1, x_2, x_3, \ldots) = \frac{1}{1 - x_1}, \\
&\text{e) } Z_{\mathcal{S}}(x_1, x_2, x_3, \ldots) = \frac{1}{(1 - x_1)(1 - x_2)(1 - x_3)\ldots}, \\
&\text{f) } Z_E(x_1, x_2, x_3, \ldots) = \exp(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots), \\
&\text{g) } Z_\varepsilon(x_1, x_2, x_3, \ldots) = x_1 \exp(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots).
\end{aligned}
\tag{1.13}
$$

The notion of cycle index series Z_F gives a simultaneous generalization of both the series $F(x)$ and $\widetilde{F}(x)$. In fact, we have the following fundamental theorem.

Theorem 1.13. *For any species of structures F, we have*

$$
\text{a) } F(x) = Z_F(x, 0, 0, \ldots), \qquad \text{b) } \widetilde{F}(x) = Z_F(x, x^2, x^3, \ldots).
$$

Proof. To establish a), proceed as follows. Substituting $x_1 = x$ and $x_i = 0$, for all $i \geq 2$, in equation (1.12) gives

$$
Z_F(x, 0, 0, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathcal{S}_n} \text{fix } F[\sigma] x^{\sigma_1} 0^{\sigma_2} 0^{\sigma_3} \ldots \right).
$$

Now for each fixed value of $n \geq 0$, $x^{\sigma_1} 0^{\sigma_2} 0^{\sigma_3} \ldots = 0$, except if $\sigma_1 = n$ and $\sigma_i = 0$ for $i \geq 2$. In other words, only the identity permutations $\sigma = \mathrm{Id}_n$ contribute to the sum. Thus

$$
\begin{aligned}
Z_F(x, 0, 0, \ldots) &= \sum_{n \geq 0} \frac{1}{n!} \mathrm{fix}\, F[\mathrm{Id}_n] x^n \\
&= \sum_{n \geq 0} \frac{1}{n!} f_n x^n \\
&= F(x),
\end{aligned}
$$

since all F-structures are fixed by transport along the identity. Equality b) is based upon a lemma of Cauchy-Frobenius (alias Burnside). Indeed, we have

$$
\begin{aligned}
Z_F(x, x^2, x^3, \ldots) &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \mathrm{fix}\, F[\sigma]\, x^{\sigma_1} x^{2\sigma_2} x^{3\sigma_3} \cdots \\
&= \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \mathrm{fix}\, F[\sigma]\, x^n \\
&= \sum_{n \geq 0} |F[n]/\!\sim|\, x^n \\
&= \widetilde{F}(x).
\end{aligned}
$$

$\square$

Example 1.14. As an illustration of Theorem 1.13, consider the case of the species $\mathcal{S}$ of permutations. It immediately follows from (1.13), e) that

$$
\begin{aligned}
Z_{\mathcal{S}}(x, 0, 0, \ldots) &= \frac{1}{1 - x} = \mathcal{S}(x), \\
Z_{\mathcal{S}}(x, x^2, x^3, \ldots) &= \frac{1}{(1 - x)(1 - x^2)(1 - x^3) \ldots} \\
&= \widetilde{\mathcal{S}}(x),
\end{aligned}
$$

in agreement with the formulas given earlier for $\mathcal{S}(x)$ and $\widetilde{\mathcal{S}}(x)$.

Remark 1.15. In examining Figures 1.10 and 1.11, one is easily convinced that the cycle type of $\mathrm{Inv}[\sigma]$ is independent of the nature of the points of U and only depends on the type of σ. This is a general phenomenon. Indeed, for all species F and all permutations σ of U, the cycle type $((F[\sigma])_1, (F[\sigma])_2, \ldots)$ of $F[\sigma]$ only depends on the cycle type $(\sigma_1, \sigma_2, \ldots)$ of σ (see Exercise 1.15). In particular, the number of fixed points of the permutation $F[\sigma]$, given by transport

$$
\mathrm{fix}\, F[\sigma] = |\mathrm{Fix}\, F[\sigma]| = (F[\sigma])_1,
$$

only depends on the numbers $\sigma_1, \sigma_2, \ldots$. Hence, in the definition of cycle index series (1.12), all permutations σ having the same cycle type $(\sigma_1, \sigma_2, \sigma_3, \ldots)$ contribute to the same monomial in the

variables $x_1, x_2, x_3, \dots$. In order to eliminate this redundancy, we regroup the monomials of the index series which correspond to each of these types. Since the number of permutations σ of n elements, of type $(n_1, n_2, n_3, \dots)$, is given by

$$\frac{n!}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots},$$

we obtain, after simplification of the $n!$, the following variant for the definition of the index series of any species F:

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n_1 + 2n_2 + 3n_3 + \dots < \infty} \text{fix } F[n_1, n_2, n_3, \dots] \frac{x_1^{n_1} \, x_2^{n_2} \, x_3^{n_3} \dots}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots}.$$

Here $\text{fix } F[n_1, n_2, n_3, \dots]$ denotes the number of F-structures on a set of $n = \sum_{i \geq 1} i n_i$ elements which are fixed under the action of any (given) permutation of type $(n_1, n_2, n_3, \dots)$. In other words, introducing the compact notations $\mathbf{n} = (n_1, n_2, n_3, \dots)$, and $\text{aut}(\mathbf{n}) = 1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots$, we have

$$\text{fix } F[\mathbf{n}] = \text{coeff}_{\mathbf{n}} Z_F := \text{aut}(\mathbf{n}) \, [x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots] \, Z_F(x_1, x_2, x_3, \dots).$$

1.2.4   Combinatorial equality

To conclude the present section, we discuss various concepts of equality which one encounters in the theory of species of structures. Strictly speaking, two species F and G are equal or identical if they have the same structures and the same transports: for all finite sets U, $F[U] = G[U]$, and for all bijections $\sigma : U \longrightarrow V$, $F[\sigma] = G[\sigma]$. However, this concept of identity is very restrictive. A much weaker version of equality between species is that of equipotence. It is obtained by replacing the set equalities $F[U] = G[U]$ by bijections $F[U] \longrightarrow G[U]$.

Definition 1.16. Let F and G be two species of structures. An **equipotence** α of F to G is a family of bijections α_U, where for each finite set U,

$$\alpha_U : F[U] \overset{\sim}{\longrightarrow} G[U].$$

The two species F and G are then called equipotent, and one writes $F \equiv G$. In other words, $F \equiv G$ if and only if there is the same number of F-structures as G-structures on all finite sets U.

For example, the species $\mathcal{S}$ of permutations is equipotent to the species L of linear orders since one has $|\mathcal{S}[U]| = |U|! = |L[U]|$ for all finite sets U. Clearly,

$$F \equiv G \quad \Leftrightarrow \quad F(x) = G(x).$$

The concept of equipotence is useful when one is only interested in the enumeration of labelled structures. However, it turns out to be inadequate when one wants to enumerate the isomorphism types of structures. Indeed,

$$F \equiv G \quad \not\Rightarrow \quad \widetilde{F}(x) = \widetilde{G}(x),$$
$$F \equiv G \quad \not\Rightarrow \quad Z_F = Z_G,$$

as has been already observed for the species $\mathcal{S}$ and L. The "good" notion of equality between species of structures lies half-way between identity and equipotence. It is the concept of isomorphism of species. It requires that the family of bijections $\alpha_U : F[U] \longrightarrow G[U]$ satisfy an additional condition relative to the transport of structures, called the **naturality condition**.

Definition 1.17. Let F and G be two species of structures. An **isomorphism** of F to G is a family of bijections $\alpha_U : F[U] \longrightarrow G[U]$ which satisfies the following naturality condition: for any bijection $\sigma : U \longrightarrow V$ between two finite sets, the following diagram commutes:

$$
\begin{array}{ccc}
F[U] & \xrightarrow{\ \alpha_U\ } & G[U] \\
{\scriptstyle F[\sigma]}\Big\downarrow & & \Big\downarrow{\scriptstyle G[\sigma]} \\
F[V] & \xrightarrow{\ \alpha_V\ } & G[V]
\end{array}
$$

In other words, for any F-structure $s \in F[U]$, one must have $\sigma \cdot \alpha_U(s) = \alpha_V(\sigma \cdot s)$. The two species F and G are then said to be isomorphic, and one writes $F \simeq G$.

Informally, the naturality condition means that, for any F-structure s on U, the corresponding G-structure $\alpha_U(s)$ on U can be described without appealing to the nature of the elements of U. Although much weaker than the concept of identity, the concept of isomorphism is nevertheless compatible with the transition to series (see Exercise 1.19) in the sense that

$$
F \simeq G \quad \Rightarrow \quad \begin{cases} F(x) = G(x), \\ \widetilde{F}(x) = \widetilde{G}(x), \\ Z_F(x_1, x_2, x_3, \ldots) = Z_G(x_1, x_2, x_3, \ldots). \end{cases}
$$

We will have many occasions to verify that two isomorphic species essentially possess the "same" combinatorial properties. Henceforth they will be considered as equal in the combinatorial algebra developed in the next sections. Thus we write $F = G$ in place of $F \simeq G$, and say that there is a **combinatorial equality** between the species F and G.

Example 1.18. There exist many classic bijections showing that the species L and $\mathcal{S}$ are equipotent. These bijections $\varphi_U : L[U] \to S[U]$ are all based on a linear order $\leq_U$ given a priori on the underlying set U. The most common, when $U = [n]$, consists of identifying the list $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ with the bijection $i \mapsto \sigma(i)$.

Example 1.19. Another classical bijection, called the **fundamental transformation** (see Foata [45] or Knuth [172]), is particularly elegant. Here is the description. Given a list

$$
\lambda = (u_1, u_2, \ldots, u_i, \ldots, u_n),
$$

in $L[U]$, let $i_1, i_2, \ldots, i_k$ be the increasing sequence of indices for which the u_{i_j} are the minimum from left to right (records) according to the order $\leq_U$. That is to say $u_{i_j} = \min\{u_i | i \leq i_j\}$, with j

running from 1 to k. In particular, $i_1 = 1$. One then defines $\tau = \varphi_U(\lambda) \in S[U]$ as being the permutation whose disjoint cycle decomposition is $\tau = (u_1, \ldots, u_{i_2-1})(u_{i_2}, \ldots, u_{i_3-1}) \ldots (u_{i_k}, \ldots, u_n)$. For example, for $\lambda = (5, 9, 7, 3, 8, 1, 4, 6, 2) \in L[9]$, the minima from left to right are $5, 3, 1$ and $i_1 = 1, i_2 = 4, i_3 = 6$, so that $\tau = \varphi_{[9]}(\lambda) = (5, 9, 7)(3, 8)(1, 4, 6, 2)$. Rewriting the cycles according to increasing order of their minimum elements, we get $\tau = (1, 4, 6, 2)\,(3, 8)\,(5, 9, 7)$. This is the so-called **standard form** for τ. Conversely, to recover λ from τ written in standard form, it suffices to write the cycles of τ in decreasing order of their minimum elements, then removing the parentheses. The fundamental transformation $\lambda \to \tau = \varphi_U(\lambda)$ has the advantage of preserving a large part of the functional digraph of these structures. It is compatible with the transport of structures along increasing bijections $\sigma : (U, \leq_U) \longrightarrow (V, \leq_V)$, in the sense that $\sigma \cdot \varphi_U(\lambda) = \varphi_V(\sigma \cdot \lambda)$. However, this is not the case for an arbitrary bijection $\sigma : U \longrightarrow V$ since the species L and $\mathcal{S}$ are not isomorphic (see Exercise 1.19, d)).

1.2.5   Contact of order n

Here is a last notion of equality, more topological but extremely useful when constructing species of structures by successive approximations. It is the concept of **contact of order** n between species of structures, for an integer $n \geq 0$. Recall that given two formal power series $a(x) = \sum_{n \geq 0} a_n x^n$ and $b(x) = \sum_{n \geq 0} b_n x^n$, one says that $a(x)$ and $b(x)$ have **contact of order** n, and one writes $a(x) =_n b(x)$, if for all $k \leq n$, $\left[x^k\right] a(x) = \left[x^k\right] b(x)$. In other words, letting $a_{\leq n}(x) = \sum_{0 \leq k \leq n} a_k x^k$, one has $a(x) =_n b(x)$ if and only if $a_{\leq n}(x) = b_{\leq n}(x)$. Contact of order n for index series is defined in a similar fashion by setting

$$h_{\leq n}(x_1, x_2, x_3, \ldots) = \sum_{n_1 + 2n_2 + 3n_3 + \ldots \leq n} h_{n_1 n_2 n_3 \ldots} x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots . \tag{1.14}$$

By analogy, one has the following definition for species of structures.

Definition 1.20. Let F and G be two species of structures and n, an integer ≥ 0. One says that F and G have **contact of order** n, and one writes $F =_n G$, if the combinatorial equality $F_{\leq n} = G_{\leq n}$ is valid, where $F_{\leq n}$ denotes the species obtained by restriction of F to sets of cardinality $\leq n$. More precisely, for finite sets U and V, and a bijection $\sigma : U \longrightarrow V$, set

$$F_{\leq n}[U] = \begin{cases} F[U], & \text{if, } |U| \leq n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Transport of structures for $F_{\leq n}$ being defined in a straightforward manner. It is clear that when species F and G have contact of order n, their associated series also have contact of order n:

$$F =_n G \implies \begin{cases} F(x) =_n G(x), \\ \widetilde{F}(x) =_n \widetilde{G}(x), \\ Z_F(x_1, x_2, x_3, \ldots) =_n Z_G(x_1, x_2, x_3, \ldots). \end{cases}$$

Definition 1.21. LIMIT OF A SEQUENCE OF SPECIES OF STRUCTURES. A sequence $(F_n)_{n\geq 0}$ of species of structures is said to **converge to** a species F, written as $\lim_{n\to\infty} F_n = F$, if for any integer $N \geq 0$, there exists $K \geq 0$ such that for all $n \geq K$, $F_n =_N F$. For any species F, it is clear that

$$\lim_{n\to\infty} F_{\leq n} = F.$$

This concept of limit is compatible with passage to the associated series (see Exercise 1.22).

1.3 Exercises

Exercises for Section 1.1

Exercise 1.1. Verify the functoriality of the transport of graphs along bijections, i.e., show that the transport functions $\mathcal{G}[\sigma]$ satisfy Equations (1.1) and (1.2).

Exercise 1.2. Let F be a species of structures and let $\sigma : U \longrightarrow V$ be any bijection between finite sets. Use functoriality to show that the transport function $F[\sigma] : F[U] \longrightarrow F[V]$ is necessarily a bijection.

   Hint: Show that $F[\sigma]^{-1} = F[\sigma^{-1}]$.

Exercise 1.3. Describe the transport functions for the following species: E, ε , X, 1, and 0.

Exercise 1.4. Show how to define, with the help of axioms, the following species: $\mathcal{G}^c$, $\mathcal{D}$, $\mathfrak{a}$, $\wp$, Par, Inv, $\mathcal{S}$, $\mathcal{C}$, L. In each case describe the transport of structures.

Exercise 1.5. Figure 1.12 describes a structure belonging to the species Cha of **chains** (non-oriented). Describe rigorously this species.
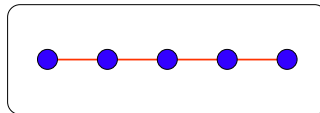

Figure 1.12: A chain.

Exercise 1.6. For all integers $n \geq 0$, designate by $\mathcal{S}_n$ the symmetric group formed of permutations (bijections) of $[n] = \{1, 2, \ldots, n\}$, under the operation of composition.

   a) Show that every species of structures F induces, for each $n \geq 0$, an action $\mathcal{S}_n \times F[n] \longrightarrow F[n]$, of the group $\mathcal{S}_n$ on the set $F[n]$ of F-structures on $[n]$, by setting $\sigma \cdot s = F[\sigma](s)$ for $\sigma \in \mathcal{S}_n$ and $s \in F[n]$.

   b) Conversely, show that any family of set actions $(\mathcal{S}_n \times F_n \longrightarrow F_n)_{n\geq 0}$, allows the definition of a species of structures F for which the families of actions above are isomorphic.

Exercise 1.7. The geometric figure 1.13 describes a structure belonging to the species Oct of **octopuses** (see [20]). Describe rigorously this species.
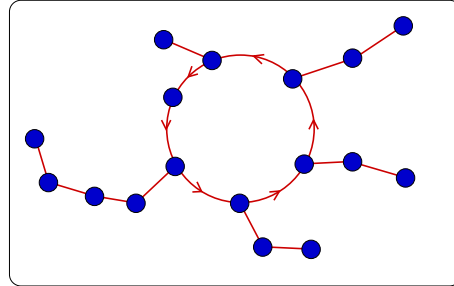

Figure 1.13: An octopus.

Exercise 1.8. Let F be a species of structures. Prove that the relation "is isomorphic to" is an equivalence relation on the totality of F-structures. More precisely, prove that for all F-structures s, t, and r, one has

a) $s \sim s$

b) $s \sim t \implies t \sim s$

c) $s \sim t$ and $t \sim r$ implies that $s \sim r$, where $s \sim t$ signifies that there exists an isomorphism from s to t.

Exercise 1.9.   a) List all trees on each of the sets: $\emptyset$, $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, and $\{1, 2, 3, 4\}$.

b) List all isomorphism types of trees (i.e.: unlabeled trees) having at most 7 vertices.

Exercise 1.10. Let L be the species of linear orders and $\mathcal{S}$ be that of permutations.

a) Show that for all $n \geq 0$ the number of L-structures on a set of cardinality n and the number of $\mathcal{S}$-structures on a set of cardinality n are both equal to $n!$.

a) Show that for all $n \geq 2$ the number of isomorphism types of L-structures on a set of cardinality n is strictly less than the number of types of S-structures on a set of cardinality n.

Exercises for Section 1.2

Exercise 1.11. Verify, by direct enumeration, formulas (1.10) for the generating series of the species L, $\mathcal{S}$, $\mathcal{C}$, E, $\in$, $\wp$, X, 1, 0, $\mathcal{G}$, $\mathcal{D}$, and End.

Exercise 1.12. Verify that the type generating series of the species Par and $\mathcal{S}$ coincide.

Exercise 1.13. Verify, by direct enumeration, formulas (1.11) for the type generating series for the species L, $\mathcal{S}$, $\mathcal{C}$, E, $\in$, $\wp$, X, 1, and 0.

Exercise 1.14. Consider two finite sets U and V such that $|U| = |V|$.

    a) Show that for all linear orders $s \in L[U]$ and $t \in L[V]$ there exists a unique bijection $\sigma : U \longrightarrow V$ such that $L[\sigma](s) = t$.

    b) Show that two permutations $\alpha \in S[U]$ and $\beta \in S[V]$ are isomorphic if and only if they have the same cycle type.

Exercise 1.15. Prove that for any species F and any permutation σ of U, the cycle type

$$((F[\sigma])_1, (F[\sigma])_2, (F[\sigma])_3, \ldots)$$

of $F[\sigma]$ only depends on the type $(\sigma_1, \sigma_2, \sigma_3, \ldots)$ of σ (and does not depend on the nature of the elements of U).

    Hint: Use the functoriality of F.

Exercise 1.16. Let $(n_1, n_2, n_3, \ldots)$ be a sequence of integers satisfying the condition $\sum_{i \geq 1} i\, n_i = n$. Prove that the number of permutations σ of type $(n_1, n_2, n_3, \ldots)$ of a set with n elements is given by the expression

$$\frac{n!}{1^{n_1} n_1! \, 2^{n_2} n_2! \, 3^{n_3} n_3! \ldots}.$$

    Hint: Show that the number of automorphisms (i.e., permutations τ such that $\sigma = \tau^{-1} \sigma \tau$) of a permutation σ of cycle type $\mathbf{n} = (n_1, n_2, n_3, \ldots)$ is aut$(\mathbf{n})$.

Exercise 1.17. Starting from the definition of cycle index series of a species, verify the formulas (1.13) for the index series of the species 0, 1, X, L, $\mathcal{S}$, E, and $\in$.

Exercise 1.18. Verify the formulas $F(x) = Z_F(x, 0, 0, \ldots)$ and $\widetilde{F}(x) = Z_F(x, x^2, x^3, \ldots)$ for the case of the following species: 0, 1, X, L, $\mathcal{S}$, E, and $\in$.

Exercise 1.19.   a) Verify that $F \equiv G$ if and only if $F(x) = G(x)$.

    b) Describe two distinct but isomorphic species.

    c) Show that $F \simeq G$ implies $F(x) = G(x)$, $\widetilde{F}(x) = \widetilde{G}(x)$, and $Z_F = Z_G$.

    d) Conclude from this that the species L and $\mathcal{S}$ are not isomorphic.

Exercise 1.20.   a) Show that the extraction of the coefficients in equations (1.7), (1.8), and (1.9) define linear transformations.

    b) Express the linear transformation defined by equation (1.9) in terms of products of differential operators.

Exercise 1.21.   a) Let $n \geq 0$ and $u(t) \in \mathbb{K}[\![t]\!]$, the ring of formal power series in t with coefficients in $\mathbb{K}$. Show that $[t^n]\, u(t) = 0$ if and only if there exists $w(t) \in \mathbb{K}[\![t]\!]$ such that

$$u(t) = t\, w'(t) - n\, w(t).$$

b) Let $n \geq k \geq 0$. Also, let $u(t)$ and $v(t)$ be in $\mathbb{K}[\![t]\!]$. Show that $n! \, [t^n] \, u(t) = k! \, [t^k] \, v(t)$ if and only if there exists a $w(t) \in \mathbb{K}[\![t]\!]$ such that

$$n_{<n-k>} \, u(t) - t^{n-k} \, v(t) = t \, w'(t) - n \, w(t),$$

where $\lambda_{<m>} := \lambda \, (\lambda - 1) \cdots (\lambda - m + 1)$, if $m > 0$, and $\lambda_{<0>} = 1$.

c) Let $n_1, n_2, \ldots, n_k$ be integers ≥ 0, and $h(x_1, x_2 \ldots, x_k) \in \mathbb{K}[\![x_1, x_2 \ldots, x_k]\!]$. Show that

$$\left[x_1^{n_1} \, x_2^{n_2} \cdots x_k^{n_k}\right] h(x_1, x_2, \ldots, x_k) = 0$$

if and only if there exists $w(x_1, \ldots, x_k) \in \mathbb{K}[\![x_1, \ldots, x_k]\!]$ such that

$$h(x_1, \ldots, x_k) = \left\{\left(x_1 \frac{\partial}{\partial x_1} - n_1\right)^2 + \ldots + \left(x_k \frac{\partial}{\partial x_k} - n_k\right)^2\right\} w(x_1, \ldots, x_k).$$

d) State and prove a necessary and sufficient condition analogous to b) for the equality

$$c_{n_1,\ldots,n_k} \left[x_1^{n_1} \, x_2^{n_2} \cdots x_k^{n_k}\right] u(x_1, x_2, \ldots, x_k) = c_{m_1,\ldots,m_k} \left[x_1^{m_1} \, x_2^{m_2} \cdots x_k^{m_k}\right] v(x_1, x_2, \ldots, x_k).$$

Exercise 1.22. We say that a sequence of formal power series $a_n(x)$ converges to a power series $a(x)$, and we write $\lim_{n\to\infty} a_n(x) = a(x)$ if for any integer $N > 0$, there exists $K > 0$ such that $n \geq K$ implies $a_n(x) =_N a(x)$.

a) Using the concept of contact of order n for the index series established in (1.14, define the notion of limit of a sequence of index series.

b) Show that for two species of structures $(F_n)_{n\geq 0}$ and F,

$$\lim_{n\to\infty} F_n = F \implies \begin{cases} \lim_{n\to\infty} F_n(x) = F(x), \\ \lim_{n\to\infty} \widetilde{F_n}(x) = \widetilde{F}(x), \\ \lim_{n\to\infty} Z_{F_n}(x_1, x_2, x_3, \ldots) = Z_F(x_1, x_2, x_3, \ldots). \end{cases}$$

Chapter 2

Operations on Species

In this chapter we describe the basic operations on species of structures. Various combinatorial operations on species of structures are used to produce new ones, in general more complex. The operations introduced here are addition, multiplication, substitution and differentiation of species of structures. They constitute a combinatorial lifting of the corresponding operations on formal power series. The problems of specification, classification and enumeration of structures are then greatly simplified, using this algebra of species. Also, this approach reveals a remarkable link between the composition of functions and the plethystic substitution of symmetric functions, in the context of Pólya theory.

Three other combinatorial operations are introduced in Sections 2.3 and 2.4, namely pointing ($^\bullet$), cartesian product ($\times$) and functorial composition ($\square$). The pointing operation interprets combinatorially the operator $x\frac{d}{dx}$. The cartesian product, $F \times G$, which consists of superimposing structures of species F and G, corresponds to coefficient-wise product of exponential generating series, known as "Hadamard product". The functorial composition, not to be confused with substitution, is a very natural operation if one recalls that a species of structures can be considered as a functor. Many varieties of graphs and multigraphs can be simply expressed with the help of this operation.

2.1   Addition and multiplication

We now introduce several operations on species of structures. There results a combinatorial algebra, allowing the construction and analysis of a multitude of species, as well as the calculation of associated series (generating series and cycle index series). These operations between species often constitute combinatorial analogs of the usual operations, addition ($+$), multiplication ($\cdot$), substitution ($\circ$) and differentiation ($'$) on their exponential generating functions.

In the algebraic context of formal power series in one variable x, given two series of exponential

Operation	Coefficient h_n
$h = f + g$	$h_n = f_n + g_n$
$h = f \cdot g$	$h_n = \sum_{i+j=n} \dfrac{n!}{i!\,j!} f_i g_j$
$h = f \circ g$ $\quad(g(0)=0)$	$h_n = \sum_{\substack{0 \leq k \leq n \\ n_1+\ldots+n_k=n}} \dfrac{n!}{k! n_1! \ldots n_k!} f_k\, g_{n_1} \cdots g_{n_k}$
$h = f'$	$h_n = f_{n+1}$

Table 2.1: Coefficients for some operation on series.

type

$$f = f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \qquad \text{and} \qquad g = g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!},$$

Table 2.1 recalls the general coefficient h_n of the series

$$h = h(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

constructed from f and g in the following cases: $h = f + g$, $h = f \cdot g$, $h = f \circ g = f(g)$, and $h = \frac{d}{dx} f = f'$. By analogy, let us now consider two species of structures F and G and consider the problem of constructing some other species, denoted by $F + G$, $F \cdot G$, $F \circ G$, and F', in order to have, for the corresponding generating series, $(F + G)(x) = F(x) + G(x)$, $(F \cdot G)(x) = F(x)\, G(x)$, $(F \circ G)(x) = F(G(x))$, and $F'(x) = \frac{d}{dx} F(x)$. These equalities between generating series signify that the new species $F + G$, $F \cdot G$, $F \circ G$ and F' should be defined so that the enumeration of their structures depends "solely" on the enumeration of the F and G-structures, via the following formulas:

1. the number of $(F + G)$-structures on n elements is $|(F + G)[n]| = |F[n]| + |G[n]|$;

2. the number of $(F \cdot G)$-structures on n elements is

$$|(F \cdot G)[n]| = \sum_{i+j=n} \frac{n!}{i!j!} |F[i]|\, |G[j]|;$$

3. the number of $(F \circ G)$-structures on n elements is

$$|(F \circ G)[n]| = \sum_{j=0}^{n} \sum_{\substack{n_1+n_2+\ldots+n_j=n \\ n_i > 0}} \frac{1}{j!} \binom{n}{n_1\, n_2 \ldots n_j} |F[j]| \prod_{i=1}^{j} |G[n_i]|;$$

4. the number of F'-structures on n elements is $|F'[n]| = |F[n+1]|$.

There could exist, a priori, many candidates for these definitions. However, there are very natural solutions which are, moreover, compatible with transport of structures. As we will see, this proves to be fundamental, in particular for the calculation of index series. We only consider, in the present section, the operations of addition and multiplication. Substitution and derivation will be treated in Section 2.2.

2.1.1 Sum of species of structures

As a motivating example, let us consider the species $\mathcal{G}^c$ of connected simple graphs and the species $\mathcal{G}^d$ of disconnected (i.e., empty or having at least two connected components) simple graphs. The evident fact that every graph is either connected or disconnected gives rise to the equality $\mathcal{G}[U] = \mathcal{G}^c[U] + \mathcal{G}^d[U]$, with "+" standing for set theoretical disjoint union. We then say that the species $\mathcal{G}$ is the sum of the species $\mathcal{G}^c$ and $\mathcal{G}^d$, and we write $\mathcal{G} = \mathcal{G}^c + \mathcal{G}^d$. This example serves as prototype for the general definition of addition of species:

Definition 2.1. Let F and G be two species of structures. The species $F + G$, called the **sum** of F and G, is defined as follows: an $(F + G)$-structure on U is an F-structure on U or (exclusive) a G-structure on U. In other words, for any finite set U, one has

$$(F + G)[U] = F[U] + G[U] \qquad \text{("+" standing for disjoint union).}$$

The transport along a bijection $\sigma : U \longrightarrow V$ is carried out by setting, for any $(F + G)$-structure s on U,

$$(F + G)[\sigma](s) = \begin{cases} F[\sigma](s), & \text{if } s \in F[U], \\ G[\sigma](s), & \text{if } s \in G[U]. \end{cases}$$

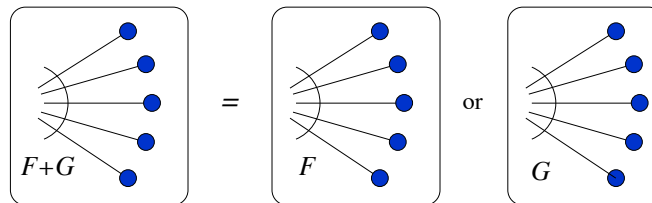In a pictorial fashion, any $(F + G)$-structure can be represented by Figure 2.1.

Figure 2.1: A typical structure of species $F + G$.

Remark 2.2. In the case where certain F-structures are also G-structures (i.e., $F[U] \cap G[U] \neq \emptyset$), one must at first form distinct copies of the sets $F[U]$ and $G[U]$. A standard way of distinguishing the F-structures from the G-structures is to replace the set $F[U]$ by the isomorphic set $F[U] \times \{1\}$ and $G[U]$ by $G[U] \times \{2\}$, and to set $(F + G)[U] := (F[U] \times \{1\}) \cup (G[U] \times \{2\})$.

The operation of addition is associative and commutative, up to isomorphism. Moreover, the empty species 0 (not having any structure: $0[U] = \emptyset$) is a neutral element for addition, i.e.: $F + 0 = 0 + F = F$, for all species F. We leave it to the reader to verify these properties as well as the following proposition.

Proposition 2.3. *Given two species of structures F and G, the associated series of the species $F + G$ satisfy the equalities*

$$
\begin{aligned}
&\text{a) } (F + G)(x) = F(x) + G(x),\\
&\text{b) } \widetilde{(F + G)}(x) = \widetilde{F}(x) + \widetilde{G}(x),\\
&\text{c) } Z_{F+G} = Z_F + Z_G.
\end{aligned}
\tag{2.1}
$$

Example 2.4. Let E_{even} (respectively E_{odd}), be the species of sets containing an even number (respectively odd number) of elements. Then $E = E_{\text{even}} + E_{\text{odd}}$ and Proposition 2.3 takes the form of the equalities (see Exercise 2.3)

a) $e^x = \cosh(x) + \sinh(x),$

b) $\dfrac{1}{(1 - x)} = \dfrac{1}{(1 - x^2)} + \dfrac{x}{(1 - x^2)},$

$\qquad(2.2)$

c) $\exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots\right) = e^{\left(\frac{x_2}{2} + \frac{x_4}{4} + \ldots\right)}\left(\cosh(x_1 + \frac{x_3}{3} + \ldots) + \sinh(x_1 + \frac{x_3}{3} + \ldots)\right).$

The operation of addition can be extended to **summable** families of species in the following sense.

Definition 2.5. A family $(F_i)_{i\in I}$ of species of structures is said to be **summable** if for any finite set U, $F_i[U] = \emptyset$, except for a finite number of indices $i \in I$. The sum of a summable family $(F_i)_{i\in I}$ is the species $\sum_{i\in I} F_i$ defined by the equalities

$$
\text{a)} \qquad \left(\sum_{i\in I} F_i\right)[U] = \sum_{i\in I} F_i[U] = \bigcup_{i\in I} F_i[U] \times \{i\},
\tag{2.3}
$$

$$
\text{b)} \qquad \left(\sum_{i\in I} F_i\right)[\sigma](s, i) = \left(F_i[\sigma](s), i\right),
\tag{2.4}
$$

where $\sigma : U \longrightarrow V$ is a bijection and $(s, i) \in (\sum_{i\in I} F_i)[U]$. We leave to the reader the task of verifying that $\sum_{i\in I} F_i$, defined in this way, is indeed a species of structures, that the families of associated series are summable (see Exercise 2), and that one has

$$
\begin{aligned}
&\text{a) } \left(\sum_{i\in I} F_i\right)(x) = \sum_{i\in I} F_i(x),\\
&\text{b) } \left(\widetilde{\sum_{i\in I} F_i}\right)(x) = \sum_{i\in I} \widetilde{F_i}(x),\\
&\text{c) } Z_{\left(\sum_{i\in I} F_i\right)} = \sum_{i\in I} Z_{F_i}.
\end{aligned}
\tag{2.5}
$$

Definition 2.6. CANONICAL DECOMPOSITION. Each species F gives rise canonically to an enumerable family $(F_n)_{n \geq 0}$ of species defined by setting, for each $n \in N$

$$F_n[U] = \begin{cases} F[U], & \text{if } |U| = n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

with the obvious induced transports. We say that F_n is the species F **restricted to cardinality** n. The family $(F_n)_{n \geq 0}$ is clearly summable and we obtain the following **canonical decomposition**:

$$F = F_0 + F_1 + F_2 + \ldots + F_n + \ldots.$$

In the case where $F = F_k$ (i.e., $F_n = 0$ for $n \neq k$), we say that F is **concentrated on the cardinality** k.

Example 2.7. Taking, for example, the species P of polygons (introduced in Section 1.1), we obtain $P = P_0 + P_1 + P_2 + \ldots + P_n + \ldots$, where P_n designates the species of all n-gons. In an analogous fashion, E_n is the species of sets of cardinality n (in particular $E_0 = 1$ and $E_1 = X$). One has the combinatorial equality

$$E = E_0 + E_1 + E_2 + \ldots + E_n + \ldots,$$

which is reflected, in terms of the associated series, by the identities

a) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!} + \ldots,$

b) $\dfrac{1}{(1-x)} = 1 + x + x^2 + \ldots + x^n + \ldots,$

c) $\exp\left(x_1 + \dfrac{x_2}{2} + \dfrac{x_3}{3} + \ldots\right) = \displaystyle\sum_{n \geq 0} \sum_{k_1 + 2k_2 + 3k_3 + \ldots = n} \dfrac{x_1^{k_1} \, x_2^{k_2} \, x_3^{k_3} \ldots}{1^{k_1} k_1! 2^{k_2} k_2! 3^{k_3} k_3! \ldots}.$

Example 2.8. Other examples of infinite sums of species are given by the formulas

$$S = \sum_{k \geq 0} S^{[k]} \qquad \text{and} \qquad \text{Par} = \sum_{k \geq 0} \text{Par}^{[k]},$$

where $S^{[k]}$ denotes the species of permutations having exactly k cycles and $\text{Par}^{[k]}$, the species of partitions having exactly k blocks (or classes).

The finite sum $F + F + \ldots + F$ of n copies of the same F is often denoted by nF. Clearly one has $(nF)(x) = nF(x)$, $\widetilde{(nF)}(x) = n\widetilde{F}(x)$, and $Z_{nF} = n \, Z_F$. The particular case where $F = 1$ (the empty set species) gives rise to the species

$$n = \underbrace{1 + 1 + \ldots + 1}_{n} = n \cdot 1$$

which possesses exactly n structures on the empty set and no structure on any set $U \neq \emptyset$. Consequently the natural numbers themselves are embedded in the combinatorial algebra of species of structures.

2.1.2   Product of species of structures

Let us examine the permutation described by Figure 2.2. We can divide this structure in two disjoint structures:

   i) a set of fixed points (those having a loop);

   ii) a derangement of the remaining elements (i.e., the permutation without fixed points formed by the non-trivial cycles).
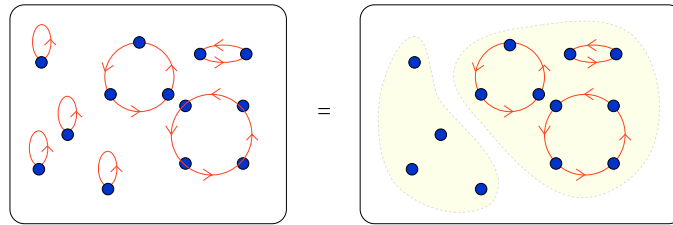

Figure 2.2: A permutation as a set of fixed points together with a derangement.

Figure 2.2 illustrates this dichotomy. An analogous decomposition clearly exists for any permutation. We say that the species $\mathcal{S}$ of permutations is the *product* of the species E of sets with the species Der of derangements and we write

$$\mathcal{S} = E \cdot \mathrm{Der}. \tag{2.6}$$

This is a typical example of the product of species of structures defined as follows.

Definition 2.9. Let F and G be two species of structures. The species $F \cdot G$ (also denoted FG), called the **product** of F and G, is defined as follows: an $(F \cdot G)$-structure on U is an ordered pair $s = (f, g)$ where

   a) f is an F-structure on a subset $U_1 \subseteq U$;

   b) g is a G-structure on a subset $U_2 \subseteq U$;

   c) (U_1, U_2) is a decomposition of U, i.e., $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \emptyset$.

In other words, for any finite set U, we have

$$(F \cdot G)[U] := \sum_{(U_1, U_2)} F[U_1] \times G[U_2],$$

with the disjoint sum being taken over all pairs (U_1, U_2) forming a decomposition of U. The transport along a bijection $\sigma : U \longrightarrow V$ is carried out by setting, for each $(F \cdot G)$-structure $s = (f, g)$ on U,

$$(F \cdot G)[\sigma](s) = \left(F[\sigma_1](f), G[\sigma_2](g) \right),$$

where $\sigma_i = \sigma\big|_{U_i}$ is the restriction of σ o U_i, $i = 1, 2$.

More informally, an $(F{\cdot}G)$-structure is an ordered pair formed by an F-structure and a G-structure over complementary disjoint subsets. A typical $(F{\cdot}G)$-structure can be represented by Figure 2.3 or by Figure 2.4.
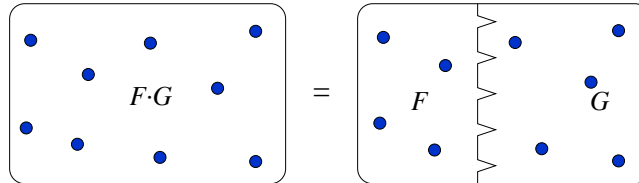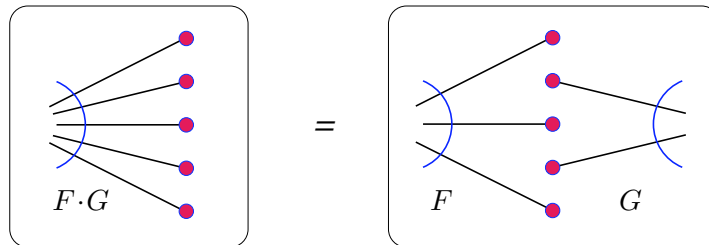

Figure 2.3: A typical product structure.

Figure 2.4: Alternate representation of a typical product structure.

The product of species is associative and commutative up to isomorphism, but in general $F{\cdot}G$ and $G{\cdot}F$ are not identical. These properties are easily established by constructing appropriate (coherent, see [230]) isomorphisms. The multiplication admits the species 1 as neutral element, and the species 0 as absorbing element, i.e.: $1{\cdot}F = F{\cdot}1 = F$ and $F{\cdot}0 = 0{\cdot}F = 0$. Moreover, multiplication distributes over addition. We leave to the reader to prove these properties as well as the following proposition (see Exercise 2.4).

Proposition 2.10. *Let F and G be two species of structures. Then the series associated with the species $F{\cdot}G$ satisfy the equalities*

$$\text{a) } (F{\cdot}G)(x) = F(x)G(x),$$

$$\text{b) } \widetilde{(F{\cdot}G)}(x) = \widetilde{F}(x)\widetilde{G}(x), \tag{2.7}$$

$$\text{c) } Z_{F{\cdot}G}(x_1, x_2, x_3, \ldots) = Z_F(x_1, x_2, x_3, \ldots)\, Z_G(x_1, x_2, x_3, \ldots).$$

Example 2.11. The preceding proposition, when applied to combinatorial Equation (2.6), yields the equalities

$$\text{a) } \frac{1}{1-x} = e^x \operatorname{Der}(x),$$

$$\text{b) } \prod_{k \geq 1} \frac{1}{1-x^k} = \frac{1}{1-x} \widetilde{\operatorname{Der}}(x),$$

c) $\displaystyle\prod_{k\geq 1}\frac{1}{1-x_k}=\exp\Big(x_1+\frac{x_2}{2}+\frac{x_3}{3}+\dots\Big)\,Z_{\mathrm{Der}}(x_1,x_2,x_3,\dots).$

One deduces (by simple division) the following expressions for the series associated to the species Der of derangements:

$$
\begin{aligned}
&\text{a) } \mathrm{Der}(x)=\frac{e^{-x}}{1-x},\\[2mm]
&\text{b) } \widetilde{\mathrm{Der}}(x)=\prod_{k\geq 2}\frac{1}{1-x^k},\\[2mm]
&\text{c) } Z_{\mathrm{Der}}(x_1,x_2,x_3,\dots)=e^{-\left(x_1+\frac{x_2}{2}+\frac{x_3}{3}+\dots\right)}\prod_{k\geq 1}\frac{1}{1-x_k}.
\end{aligned}
\qquad (2.8)
$$

Note that the classical formula

$$
d_n=n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\dots+\frac{(-1)^n}{n!}\right),
$$

giving the number of derangements of a set of n elements, is directly obtained from (2.8), a) by explicitly carrying out the product

$$
\sum_{n\geq 0}d_n\frac{x^n}{n!}=\left(\sum_{i\geq 0}(-1)^i\frac{x^i}{i!}\right)\left(\sum_{j\geq 0}x^j\right).
$$

The reader can calculate explicitly, starting from (2.8), c), the numbers $\operatorname{fix}\mathrm{Der}[n_1,n_2,\dots]$, coefficients of the index series Z_{Der} (see Exercise 2.6). As we see, the simple combinatorial equality $S=E{\cdot}\mathrm{Der}$ contains structural information which goes well beyond the simple enumeration of the labelled structures.

It is interesting to observe that the species $F+F+\dots+F$ (n terms), which is denoted $n\,F$, is also the product of the species n with the species F. This is to say $n\,F=n{\cdot}F$. Once more, this justifies identifying the integer n with the species n. For the species $\wp$ of subsets of a set, introduced in Section 1.1.2, we have the combinatorial equality $\wp=E{\cdot}E$. Translating this equality into series, we recover the equalities

$$
\wp(x)=e^x\,e^x=e^{2x},\quad \widetilde{\wp}(x)=\frac{1}{(1-x)^2},
$$

mentioned in Section 1.2. For the index series $Z_\wp$, we also immediately obtain

$$
Z_\wp(x_1,x_2,x_3,\dots)=\left(\exp\Big(x_1+\frac{x_2}{2}+\frac{x_3}{3}+\dots\Big)\right)^2,
$$

and deduce the expression $\operatorname{fix}\wp[n_1,n_2,\dots]=2^{n_1+n_2+\dots}$, which can also be obtained by a direct combinatorial argument. Similarly, the species $\wp^{[k]}$, of subsets of cardinality k, satisfies the combinatorial equality $\wp^{[k]}=E_k{\cdot}E$, where E_k denotes the species of sets of cardinality k. A simple passage to the associated series yields the equalities

a) $\wp^{[k]}(x) = e^x \dfrac{x^k}{k!}$,

b) $\widetilde{\wp^{[k]}}(k) = \dfrac{x^k}{1-x}$,

c) $Z_{\wp^{[k]}}(x_1, x_2, x_3, \ldots) = \exp\!\left(x_1 + \dfrac{x_2}{2} + \dfrac{x_3}{3} + \ldots\right) \displaystyle\sum_{n_1+2n_2+\ldots=k} \dfrac{x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots}{1^{n_1} n_1! \, 2^{n_2} n_2! \, 3^{n_3} n_3! \ldots}.$

The equality $\wp^{[k]}(x) = e^x \frac{x^k}{k!}$ gives the well-known combinatorial interpretation of binomial coefficients, $|\wp^{[k]}[n]| = \binom{n}{k}$, as the number of k-element subsets of a n-element set. Moreover, the explicit formula for the numbers fix $\wp^{[k]}[n_1, n_2, n_3 \ldots]$ given in Exercise 2.7 constitutes a generalization of the notion of binomial coefficients. Of course, we also have the combinatorial equality

$$\sum_{k \geq 0} \wp^{[k]} = \wp = E^2,$$

which, by passing to generating series, gives the identity $\sum_{k \geq 0} \binom{n}{k} = 2^n$. In virtue of associativity, the operation of multiplication can be extended to finite families F_i of species, $i = 1, \ldots, k$, by defining the product $F_1 \cdot F_2 \cdot \ldots \cdot F_k$ by

$$(F_1 \cdot F_2 \cdot \ldots \cdot F_k)[U] = \sum_{U_1 + U_2 + \ldots + U_k = U} F_1[U_1] \times F_2[U_2] \times \ldots \times F_k[U_k].$$

Here the (disjoint) sum is taken over all families $(U_i)_{1 \leq i \leq k}$ of pairwise disjoint subsets of U whose union is U. The transport along a bijection $\sigma : U \longrightarrow V$ is defined in the following **component-wise** manner: for $s_i \in F_i[U_i]$, $\quad i = 1, \ldots, k$, set

$$(F_1 \cdot F_2 \cdot \ldots \cdot F_k)[\sigma]\left((s_i)_{1 \leq i \leq k}\right) = (F_i[\sigma_i](s_i))_{1 \leq i \leq k},$$

where $\sigma_i = \sigma|_{U_i}$ denotes the restriction of σ to U_i. The product $\prod_{i \in I} F_i$ of an infinite family of species $(F_i)_{i \in I}$ can also be defined provided that this family is **multiplicable** (see Exercise 2.9). An example of a multiplicable infinite family is given in Example 2.11. In the case of a finite family $(F_i)_{1 \leq i \leq k}$ where all of its members are equal to the same species F, the product $F \cdot F \cdot \ldots \cdot F$ (k factors) is denoted F^k. An F^k-structure on a set U is therefore a k-tuple $(s_1, s_2, \ldots, s_k)$ of disjoint F-structures whose union of underlying sets is U.

Example 2.12. Taking $F = E_+$, the species of non-empty sets, we obtain the species $\mathrm{Bal}^{[k]} = (E_+)^k$ of all ballots having k levels (i.e., ordered partitions having k blocks; see Figure 2.5). We therefore have

a) $\mathrm{Bal}^{[k]}(x) = (e^x - 1)^k$,

b) $\widetilde{\mathrm{Bal}^{[k]}}(x) = \left(\dfrac{x}{1-x}\right)^k$, $\hspace{4cm}$ (2.9)

c) $Z_{\mathrm{Bal}^{[k]}}(x_1, x_2, x_3, \ldots) = \left(\exp\!\left(x_1 + \dfrac{x_2}{2} + \dfrac{x_3}{3} + \ldots\right) - 1\right)^k.$

Since the family $(E_+)^k$, $k = 0, 1, 2, \ldots$, is summable, one obtains by summation (see also Section 1.1.5) the species Bal of all ballots (independent of the number of levels):

$$\text{Bal} = \sum_{k \geq 0} \text{Bal}^{[k]} = \sum_{k \geq 0} (E_+)^k.$$

A simple summation of the associated series gives

$$
\begin{aligned}
&\text{a) } \text{Bal}(x) = \frac{1}{2 - e^x}, \\
&\text{b) } \widetilde{\text{Bal}}(x) = \frac{1 - x}{1 - 2x}, \\
&\text{c) } Z_{\text{Bal}}(x_1, x_2, x_3, \ldots) = \frac{1}{2 - \exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots\right)}.
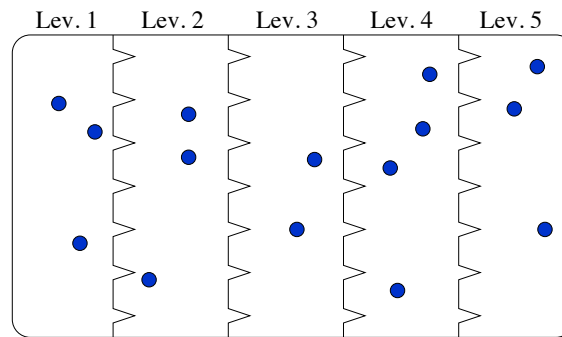\end{aligned}
\tag{2.10}
$$

Figure 2.5: A Bal[5]-structure.

Example 2.13. Consider the species L of linear orderings and its restriction L_k to sets of length k. One has the combinatorial equalities

$$
\begin{aligned}
&\text{a)} \qquad L_k = X^k, \qquad k = 0, 1, 2, \ldots; \\
&\text{b)} \qquad L = 1 + XL = \sum_{k \geq 0} X^k = \prod_{i \geq 0} \left(1 + X^{2^i}\right),
\end{aligned}
$$

where X denotes the species of singletons. For a definition of infinite product of species, see Exercise 2.9.

2.2   Substitution and differentiation

2.2.1   Substitution of species of structures

As a motivating example, let us consider an endofunction $\varphi \in \text{End}[U]$ of a set U, determined by its functional digraph, such as that of Figure 2.6 a). Two kinds of points (elements of U) can be

distinguished

i) the recurrent points, i.e., those $x \in U$ for which there exists a $k > 0$ such that $\varphi^k(x) = x$; these are the elements located on cycles;

ii) the non-recurrent points, i.e., those x for which $\varphi^k(x) \neq x$ for all $k > 0$.
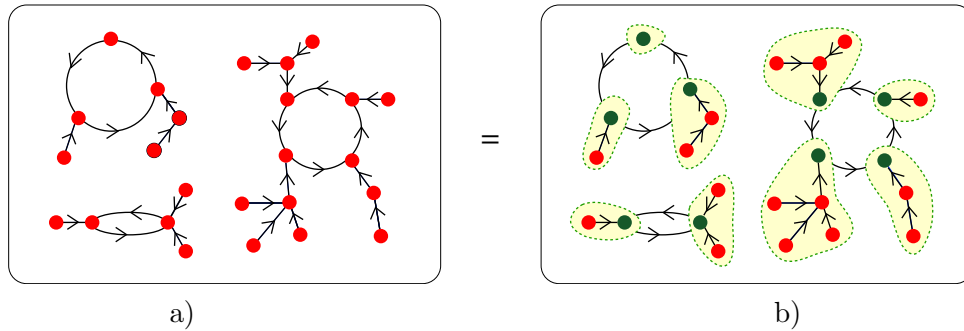

a)   b)

Figure 2.6: An endofunction as a permutation of trees.

Figure 2.6 b) shows how the endofunction φ can naturally be identified with a permutation of disjoint rooted trees. The naturality originates from the fact that we need not use the specific nature of the underlying points in order to pass from Figure 2.6 a) to Figure 2.6 b). Clearly, such an analysis can be carried out no matter which endofunction is given. Thus, every End-structure can naturally be identified with an $\mathcal{S}$-structure placed on a set of disjoint $\mathcal{A}$-structures, where, as previously, End denotes the species of endofunctions, $\mathcal{S}$, the species of permutations and $\mathcal{A}$, the species of rooted trees. In a more concise manner, we say that every End-structure is an $\mathcal{S}$-assembly of $\mathcal{A}$-structures. This situation is summarized by the combinatorial equation

$$\text{End} = \mathcal{S} \circ \mathcal{A}, \quad \text{or} \quad \text{End} = \mathcal{S}(\mathcal{A}).$$

This simple equality expresses the fact that every endofunction is essentially a permutation of (disjoint) rooted trees. It is a typical example of substitution of species, also called the (partitional) composition of species, which can be defined in general as follows.

Definition 2.14. Let F and G be two species of structures such that $G[\emptyset] = \emptyset$ (i.e., there is no G-structure on the empty set). The species $F \circ G$, also denoted $F(G)$, called the (partitional) **composite** of G in F, is defined as follows: an $(F \circ G)$-structure on U is a triplet $s = (\pi, \varphi, \gamma)$, where

i) π is a partition of U,

ii) φ is an F-structure on the set of classes of π,

iii) $\gamma = (\gamma_p)_{p \in \pi}$, where for each class p of π, γ_p is a G-structure on p.

In other words, for any finite set U, one has

$$(F \circ G)[U] = \sum_{\pi \text{ partition of } U} F[\pi] \times \prod_{p \in \pi} G[p], \tag{2.11}$$

the (disjoint) sum being taken over the set of partitions π of U (i.e., $\pi \in \mathrm{Par}[U]$). The transport along a bijection $\sigma : U \longrightarrow V$ is carried out by setting, for any $(F \circ G)$-structure $s = (\pi, \varphi, (\gamma_p)_{p \in \pi})$ on U,

$$(F \circ G)[\sigma](s) = \left(\overline{\pi}, \overline{\varphi}, (\overline{\gamma_{\overline{p}}})_{\overline{p} \in \overline{\pi}}\right), \tag{2.12}$$

where

i) $\overline{\pi}$ is the partition of V obtained by transport of π along σ,

ii) for each $\overline{p} = \sigma(p) \in \overline{\pi}$, the structure $\overline{\gamma_{\overline{p}}}$ is obtained from the structure γ_p by G-transport along $\sigma \mid_p$,

iii) the structure $\overline{\varphi}$ is obtained from the structure φ by F-transport along the bijection $\overline{\sigma}$ induced on π by σ.

In a more visual fashion, we say that an $(F \circ G)$-structure is an F-**assembly** of (disjoint) G-structures. Figures 2.7 and 2.8 give alternate graphical illustrations of this concept. The proof that $F \circ G$ as defined is a species of structures (i.e., that the transports satisfy the functoriality properties) is left to the reader. When F is the species of sets, an $(E \circ G)$-structure is more simply
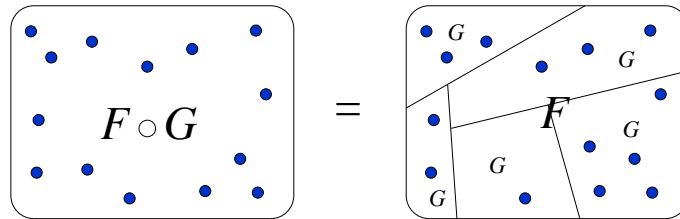

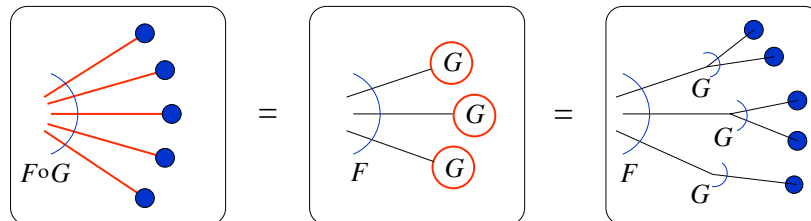Figure 2.7: An F-assembly of G-structures.

Figure 2.8: An F of G structure.

called an **assembly** of G-structures. The passage from $F \circ G$ to its generating and cycle index

series is more delicate to analyze than in the case of sum and product of species. In fact, although the formula $(F \circ G)(x) = F(G(x))$ is valid, the corresponding identity does not hold in general for unlabeled enumeration $\widetilde{(F \circ G)}(x) \neq \widetilde{F}(\widetilde{G}(x))$, as shown in Exercise 2.16. This is one of the reasons for which the introduction of cycle index series is necessary, as shown by the following result. A complete proof is given in Chapter 4 of [22].

Theorem 2.15. *Let F and G be two species of structures and suppose that $G[\emptyset] = \emptyset$. Then the series associated to the species $F \circ G$ satisfy the equalities*

a) $(F \circ G)(x) = F(G(x))$,

b) $\widetilde{(F \circ G)}(x) = Z_F\left(\widetilde{G}(x), \widetilde{G}(x^2), \widetilde{G}(x^3), \ldots\right)$, $\qquad(2.13)$

c) $Z_{F \circ G}(x_1, x_2, x_3, \ldots) = Z_F\left(Z_G(x_1, x_2, \ldots), Z_G(x_2, x_4, \ldots), \ldots\right)$.

The index series given in the last formula is called the **plethystic substitution** of Z_G in Z_F, and is denoted by $Z_F \circ Z_G$ (or $Z_F(Z_G)$).

Definition 2.16. Let $f = f(x_1, x_2, x_3, \ldots)$ and $g = g(x_1, x_2, x_3, \ldots)$ be two formal power series. Then the **plethystic substitution** $f \circ g$ is defined by $(f \circ g)(x_1, x_2, x_3, \ldots) = f(g_1, g_2, g_3, \ldots)$, where the following notational convention is used:

$$g_k = g(x_k, x_{2k}, x_{3k}, \ldots), \quad k = 1, 2, 3, \ldots, \qquad(2.14)$$

i.e., the power series g_k is obtained by multiplying by k the index of each variable appearing in g. Observe that $g_k = x_k \circ g = g \circ x_k$.

Example 2.17. From the combinatorial equation $\mathrm{End} = \mathcal{S} \circ \mathcal{A}$, one immediately deduces the formulas

a) $\mathrm{End}(x) = (\mathcal{S} \circ \mathcal{A})(x) = \mathcal{S}(\mathcal{A}(x)) = \dfrac{1}{1 - \mathcal{A}(x)}$,

b) $\widetilde{\mathrm{End}}(x) = \widetilde{(\mathcal{S} \circ \mathcal{A})}(x) = Z_{\mathcal{S}}\left(\widetilde{\mathcal{A}}(x), \widetilde{\mathcal{A}}(x^2), \widetilde{\mathcal{A}}(x^3), \ldots\right) = \dfrac{1}{(1 - \widetilde{\mathcal{A}}(x))(1 - \widetilde{\mathcal{A}}(x^2))(1 - \widetilde{\mathcal{A}}(x^3)) \ldots}$,

c) $Z_{\mathrm{End}} = (Z_{\mathcal{S}} \circ Z_{\mathcal{A}})(x_1, x_2, x_3, \ldots) = (1 - x_1 \circ Z_{\mathcal{A}})^{-1}(1 - x_2 \circ Z_{\mathcal{A}})^{-1}(1 - x_3 \circ Z_{\mathcal{A}})^{-1} \cdots$,

which relate the series $\mathrm{End}(x)$, $\widetilde{\mathrm{End}}(x)$ and Z_{End} to the series $\mathcal{A}(x)$, $\widetilde{\mathcal{A}}(x)$ and $Z_{\mathcal{A}}$. These series are studied more deeply in Chapter 3 of [22]. Figure 2.9 shows that the species $\mathcal{A}$ of rooted trees satisfies the combinatorial equation $\mathcal{A} = X \cdot E(\mathcal{A})$, where E designates the species of sets. Passing to series gives the formulas

a) $\mathcal{A}(x) = xe^{\mathcal{A}(x)}$,

b) $\widetilde{\mathcal{A}}(x) = x \exp\left(\widetilde{\mathcal{A}}(x) + \dfrac{\widetilde{\mathcal{A}}(x^2)}{2} + \dfrac{\widetilde{\mathcal{A}}(x^3)}{3} + \ldots\right)$, $\qquad(2.15)$

c) $Z_{\mathcal{A}}(x_1, x_2, \ldots) = x_1 \exp\left(Z_{\mathcal{A}}(x_1, x_2, \ldots) + \dfrac{1}{2} Z_{\mathcal{A}}(x_2, x_4, \ldots) + \ldots\right)$.

It is shown in Chapter 3 of [22] how these formulas allow recursive, and even explicit, calculation of the series $\mathcal{A}(x)$, $\widetilde{\mathcal{A}}(x)$ and $Z_{\mathcal{A}}(x_1, x_2, x_3, \ldots)$.
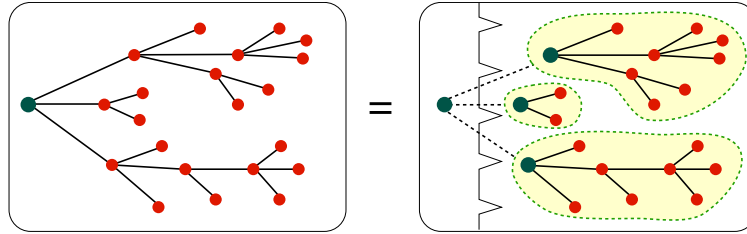
Figure 2.9: A rooted tree is a set of rooted trees attached to a root.

Example 2.18. Figure 2.10 shows that the species Bal of ballots satisfies the combinatorial equation

$$\text{Bal} = L \circ E_+, \tag{2.16}$$

where L is the species of linear orderings and E_+ that of non-empty sets. The identities (2.10) for the series associated to the species Bal can be deduced directly from this observation. Consider
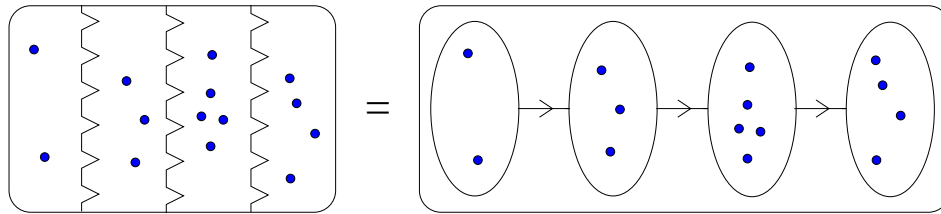

Figure 2.10: A ballot is a list of parts.

Par, the species of **partitions**. Since every partition is naturally identified to a set of non-empty disjoint sets (see Figure 2.11), we obtain the combinatorial equation

$$\text{Par} = E(E_+). \tag{2.17}$$

The following formulas are then immediately deduced.

a) $\text{Par}(x) = e^{e^x - 1}$,

b) $\widetilde{\text{Par}}(x) = \prod_{k \geq 1} \frac{1}{1 - x^k}$,

c) $Z_{\text{Par}}(x_1, x_2, x_3, \ldots) = \exp \sum_{k \geq 1} \frac{1}{k} \left(\exp\left(x_k + \frac{x_{2k}}{2} + \frac{x_{3k}}{3} + \ldots\right) - 1 \right).$

$$\tag{2.18}$$

Example 2.19. In a similar fashion, since every permutation is a set of disjoint cycles (see Figure 2.12), we have the combinatorial equation $\mathcal{S} = E \circ \mathcal{C}$, where $\mathcal{C}$ is the species of cycles (cyclic permutations). It follows that $\mathcal{S}(x) = \frac{1}{1-x} = e^{\mathcal{C}(x)}$ and we recover $\mathcal{C}(x) = \log \frac{1}{1-x}$. Moreover, we
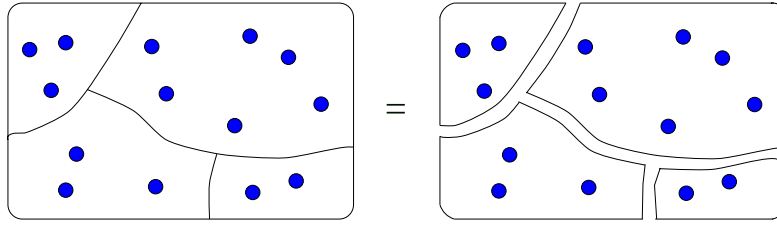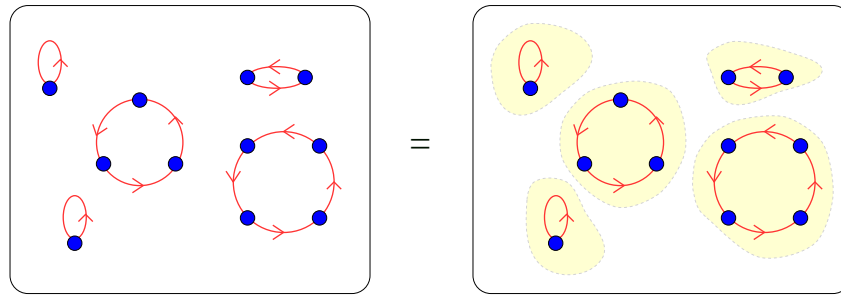
Figure 2.11: A partition is a set of parts.

Figure 2.12: A permutation is a set of cycles.

have the remarkable identities

$$
\text{a) } \prod_{k \geq 1} \frac{1}{1 - x^k} = \widetilde{\mathcal{S}}(x) = Z_E\left(\widetilde{\mathcal{C}}(x), \widetilde{\mathcal{C}}(x^2), \ldots\right) = \exp\left(\sum_{n \geq 1} \frac{1}{k} \frac{x^k}{1 - x^k}\right),
$$

$$
\text{b) } \frac{1}{1 - x_1} \frac{1}{1 - x_2} \frac{1}{1 - x_3} \cdots = Z_{\mathcal{S}}(x_1, x_2, x_3, \ldots) = \exp\left(\sum_{k \geq 1} \frac{1}{k} Z_{\mathcal{C}}(x_k, x_{2k}, x_{3k}, \ldots)\right). \tag{2.19}
$$

This last identity permits the explicit calculation of the index series $Z_{\mathcal{C}}$ of the species of cycles (see Exercise 2.21):

$$
Z_{\mathcal{C}}(x_1, x_2, x_3, \ldots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1 - x_k}, \tag{2.20}
$$

where ϕ denotes the arithmetic Euler ϕ-function.

Example 2.20. The species $\mathcal{G}$ of graphs is related to the species $\mathcal{G}^c$ of connected graphs by the combinatorial equation $\mathcal{G} = E(\mathcal{G}^c)$, since every graph is an assembly of connected graphs.

More generally, if two species F and F^c are related by a combinatorial equation of the form $F = E(F^c)$, we say that F^c is the species of **connected F-structures**. We then have

a) $F(x) = e^{F^c(x)}$,

b) $\widetilde{F}(x) = \exp \sum_{k \geq 1} \frac{1}{k} \widetilde{F^c}(x^k),$

c) $Z_F(x_1, x_2, x_3, \ldots) = \exp \sum_{k \geq 1} \frac{1}{k} Z_{F^c}(x_k, x_{2k}, x_{3k}, \ldots).$

It is interesting to note that we can also express the series $F^c(x)$, $\widetilde{F^c}(x)$ and Z_{F^c} as functions of the series $F(x)$, $\widetilde{F}(x)$ and Z_F (see Exercise 2.22).

The species X of singletons is the **neutral element** for the substitution of species: $F = F(X) = F \circ X = X \circ F = X(F)$. Substitution is associative (up to isomorphism of species). For any species of structures G, the condition $G[\emptyset] = \emptyset$ is equivalent to $G(0) = 0$. If $G(0) = 0$, one recursively defines the successive iterates $G^{\langle n \rangle}$ of G by the recursive scheme

$$G^{\langle 0 \rangle} = X, \qquad \text{and} \qquad G^{\langle n+1 \rangle} = G \circ G^{\langle n \rangle} \quad (= G^{\langle n \rangle} \circ G).$$

Example 2.21. Consider the species Preo of all the preorders (i.e., reflexive and transitive relations) and the species Ord of all order relations (i.e., antisymmetric preorders). Since every preorder "$\prec$" induces, in a natural manner, an order on an appropriate quotient set (see Figure 2.13), we obtain the combinatorial equation $\text{Preo} = \text{Ord}(E_+)$. The computation of the power series $\text{Preo}(x)$, $\text{Ord}(x)$, $\widetilde{\text{Preo}}(x)$, $\widetilde{\text{Ord}}(x)$, Z_{Preo} and Z_{Ord} is an open problem. Nevertheless, Theorem 2 implies the following relations:

a) $\text{Preo}(x) = \text{Ord}(e^x - 1),$

b) $\widetilde{\text{Preo}}(x) = Z_{\text{Ord}} \left(\dfrac{x}{1-x}, \dfrac{x^2}{1-x^2}, \dfrac{x^3}{1-x^3}, \ldots \right),$

c) $Z_{\text{Preo}}(x_1, x_2, x_3, \ldots) = Z_{\text{Ord}} \left(e^{x_1 + \frac{x_2}{2} + \cdots} - 1, e^{x_2 + \frac{x_4}{2} + \cdots} - 1, \ldots \right).$
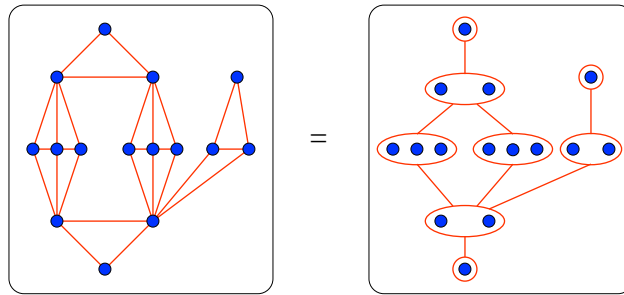

Figure 2.13:

2.2.2 The derivative of a species of structures

Given an arbitrary species of structures F, we propose to construct another species G so that their respective generating series satisfy $G(x) = \frac{d}{dx} F(x)$. This is equivalent to requiring that $|G[n]| = |F[n+1]|$, $n = 0, 1, 2, \ldots$. Hence the number of G-structures on an arbitrary finite set U should be equal to the number of F-structures on the set U to which a "new" element has been added. This suggests the following definition:

Definition 2.22. Let F be a species of structures. The species F' (also denoted by $\frac{d}{dX} F(X)$), called the **derivative** of F, is defined as follows: an F'-structure on U is an F-structure on $U^+ = U \cup \{*\}$, where $* = *_U$ is a element chosen outside of U. In other words, for any finite set U, one sets $F'[U] = F[U^+]$, where $U^+ = U + \{*\}$. The transport along a bijection $\sigma : U \longrightarrow V$ is simply carried out by setting, for any F'-structure s on U, $F'[\sigma](s) = F[\sigma^+](s)$, where $\sigma^+ : U + \{*\} \longrightarrow V + \{*\}$ is the canonical extension of σ obtained by setting $\sigma^+(u) = \sigma(u)$, if $u \in U$, and $\sigma^+(*) = *$. Figures 2.14 and 2.15 illustrate graphically the concept of F'-structure.
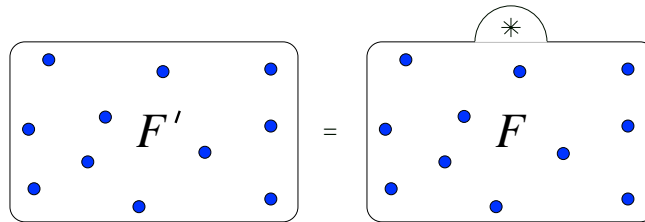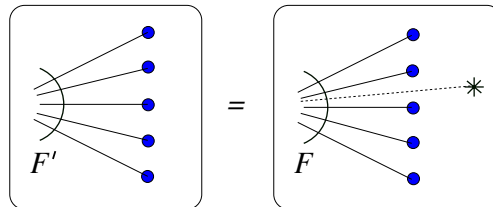

Figure 2.14: A typical structure of species F'.

Figure 2.15: Alternate representation of a typical structure of species F'

Remark 2.23. Observe that the supplementary element $*$ is not a member of the underlying set of the F'-structure on U. Also note that the element $*$ has been placed in an arbitrary position in Figure 2.15 to emphasize that the set $U + \{*\}$ on which the F-structure is constructed is not otherwise structured. The careful reader may ask himself how does one systematically (and canonically) choose a element $* = *_U$ outside each given set U. Exercise 2.29 describes a classic solution to this problem.

Example 2.24. As a standard illustration, we analyze the derivative $\mathcal{C}'$ of the species $\mathcal{C}$ of cyclic permutations. By definition, a $\mathcal{C}'$-structure on the set $U = \{a, b, c, d, e\}$ is a $\mathcal{C}$-structure on $U + \{*\}$.

It is identified in a natural manner (forgetting $*$) with a linear ordering placed on U (see Figure 2.16). In other words, we have shown the combinatorial equation $\mathcal{C}' = L$. Passing to generating series yields $\mathcal{C}'(x) = L(x) = \frac{1}{1-x}$, and, by integration,

$$\mathcal{C}(x) = \int_0^x \frac{dx}{1-x} = \log \frac{1}{1-x},$$

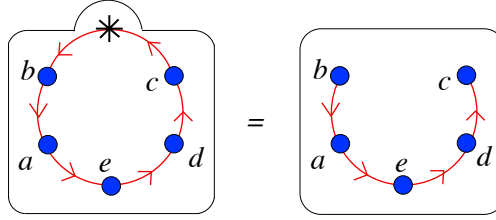which gives a third way to obtain the series $\mathcal{C}(x)$.

Figure 2.16: Breaking off a cycle at the special point gives a list.

The relation between the derivative of species and the corresponding series is summarized in the following proposition.

Proposition 2.25. *Let F be a species of structures. One has the equalities*

a) $F'(x) = \dfrac{d}{dx} F(x),$

b) $\widetilde{F'}(x) = \left(\dfrac{\partial}{\partial x_1} Z_F \right)(x, x^2, x^3, \ldots),$

c) $Z_{F'}(x_1, x_2, x_3, \ldots) = \left(\dfrac{\partial}{\partial x_1} Z_F \right)(x_1, x_2, x_3, \ldots).$

Example 2.26. Consider the species Par', derivative of the species Par of partitions. Figure 2.17 shows that a Par'-structure on a set U can be identified in a natural way to a **partial partition** on U, that is to say, a partition on a part V of U: simply take $V = U \setminus W$, where W is the class containing $*$. Let Par_P be the species of partial partitions. We then have the combinatorial equation $\mathrm{Par}_P = \mathrm{Par}'$. Applying the preceding proposition to the known series for the species Par yields

$$\begin{aligned} &\text{a) } \mathrm{Par}_P(x) = e^{x+e^x-1}, \\ &\text{b) } Z_{\mathrm{Par}_P}(x_1, x_2, \ldots) = \exp \sum_{k \geq 1} \tfrac{1}{k} \left(x_k + \exp(x_k + \tfrac{x_{2k}}{2} + \ldots) - 1 \right). \end{aligned} \qquad (2.21)$$

In particular, letting $x_i := x^i$, we obtain

$$\widetilde{\mathrm{Par}_P}(x) = \left(\frac{1}{1-x} \right)^2 \prod_{k \geq 2} \frac{1}{1-x^k}.$$
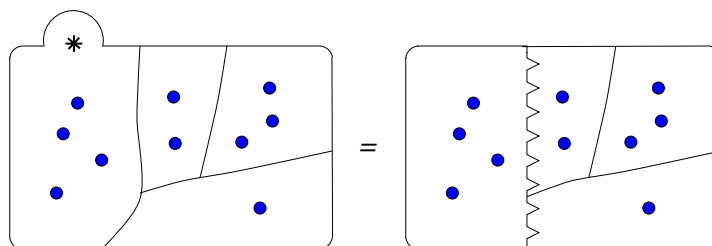
Figure 2.17: Partition with a privileged part.

Note that Par_P also satisfies the combinatorial equation $\mathrm{Par}_P = E \cdot \mathrm{Par}$, as is shown in the same Figure 2.17. This allows for a calculation of the series associated to Par_P in a different fashion. For example,

$$\mathrm{Par}_P(x) = (E \cdot \mathrm{Par})(x) = E(x)\mathrm{Par}(x) = e^x \mathrm{Par}(x),$$

agreeing with (2.21) a) obtained earlier.

Example 2.27. The derivative E' of the species E of sets satisfies the combinatorial equation $E' = E$. This constitutes a combinatorial version of the classic equality $\frac{d}{dx}e^x = e^x$. For the species L of linear orderings, Figure 2.18 shows that $L' = L^2 \quad (= L \cdot L)$, reflecting combinatorially the series identity

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \left(\frac{1}{1-x}\right)^2.$$
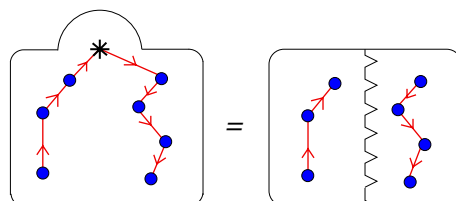
Figure 2.18: Cutting up a list at the special point.

The operation of differentiation can be iterated. For $F'' = (F')'$, we simply add successively two distinct elements, $*_1$ and $*_2$, to the underlying set. For example, we have the combinatorial equation $\mathcal{C}'' = L^2$. More generally, we set $F^{(0)} = F$ and $F^{(k)} = (F^{(k-1)})'$ when $k \geq 1$. An $F^{(k)}$-structure on U is then equivalent to an F-structure on $U \cup \{*_1, *_2, \ldots, *_k\}$, where $*_i$, $1 \leq i \leq k$, is an ordered sequence of k additional distinct elements (see Figure 2.19, where $k = 5$).

Example 2.28. Consider the species $\mathfrak{a}$ of trees. Figure 2.20 immediately shows that we have the species identity $\mathfrak{a}' = \mathcal{F} = E(\mathcal{A})$, where $\mathcal{F}$ is the species of **forests** of rooted trees (i.e., disjoint sets of rooted trees). One can then assert, a priori, that the series for $\mathfrak{a}$ and $\mathcal{F}$ are related by the following equalities.

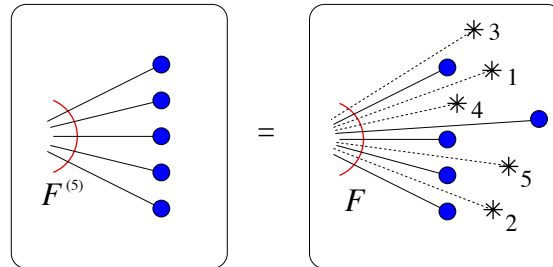Figure 2.19: Higher order derivatives of a species.

a) $\mathcal{F}(x) = \dfrac{d}{dx}\mathfrak{a}(x)$,

b) $\widetilde{\mathcal{F}}(x) = \left(\dfrac{\partial}{\partial x_1}Z_\mathfrak{a}\right)(x, x^2, x^3, \ldots)$,

c) $Z_\mathcal{F}(x_1, x_2, x_3, \ldots) = \left(\dfrac{\partial}{\partial x_1}Z_\mathfrak{a}\right)(x_1, x_2, x_3, \ldots)$.
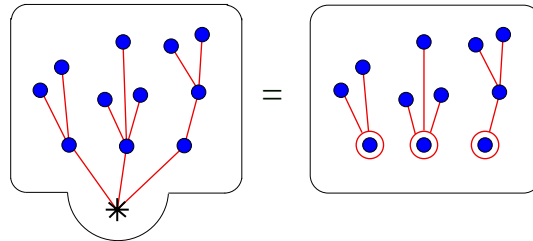
Figure 2.20: A tree with a special point decomposes as a forest.

Remark 2.29. To underline how the combinatorial differential calculus of species agrees with the classical differential calculus of formal power series, we mention that the chain rule admits the combinatorial equivalent $(F \circ G)' = (F' \circ G) \cdot G'$, where G is a species such that $G(0) = 0$ (i.e., $G[\emptyset] = \emptyset$). Consideration of Figure 2.21 suffices to show the validity of this formula. It is easily verified that the other usual rules $(F + G)' = F' + G'$, and $(F \cdot G)' = F' \cdot G + F \cdot G'$ are also satisfied in the context of species.

Nevertheless, one must be prudent when establishing analogies with classical differential calculus. For instance, although the differential equation $y' = f(x)$ with initial condition $y(0) = 0$, always has a unique solution in the setting of formal power series, one can show that the analogous equation $Y' = F(X)$, with $Y(0) = 0$, can have many non-isomorphic solutions in the algebra of species of structures (see Exercise 2.37). On the other hand, the equation $Y' = XE_3(X)$, with $Y(0) = 0$, has no species of structures solution.

Figure 2.21: Combinatorial chain rule.

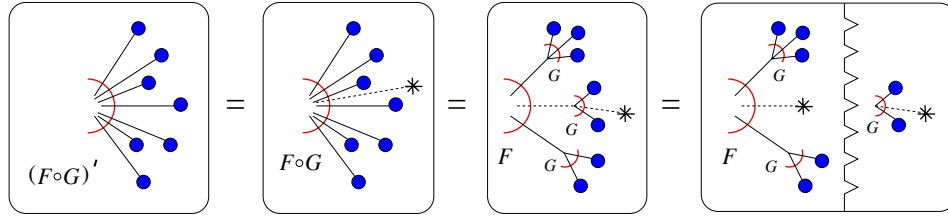**Remark 2.30.** For other variants of the theory of the species of structures (for example that of $\mathbb{L}$-species introduced in Chapter 5 of [22]) the existence and uniqueness of solutions of combinatorial differential equations coincides with the formal power series setting.

2.3 Pointing and Cartesian product

2.3.1 Pointing in a species of structures

Pointing corresponds at the combinatorial level to the differential operator $x\frac{d}{dx}$, whose effect on formal power series is:

$$x\frac{d}{dx}\sum_{n\geq 0}f_n\frac{x^n}{n!}=\sum_{n\geq 0}n\,f_n\frac{x^n}{n!}.$$

Definition 2.31. Let F be a species of structures. The species $F^\bullet$, called F dot, is defined as follows: an $F^\bullet$-structure on U is a pair $s=(f,u)$, where

   i) f is an F-structure on U,

   ii) $u\in U$ (a distinguished element).

The pair (f,u) is called a **pointed** F-structure (pointed at the distinguished element u). In other words, for any finite set U, $F^\bullet[U]=F[U]\times U$ (set-theoretic Cartesian product). The transport along a bijection $\sigma:U\longrightarrow V$ is carried out by setting $F^\bullet[\sigma](s)=(F[\sigma](f),\sigma(u))$, for any $F^\bullet$-structure $s=(f,u)$ on U. A typical $F^\bullet$-structure can be represented graphically by circling the pointed element (see Figure 2.22). The enumeration of the $F^\bullet$-structures satisfies $|F^\bullet[n]|=n\,|F[n]|$, for all $n\geq 0$.

Example 2.32. As a first illustration, let us point the species $\mathfrak{a}$ of trees. We then obtain the species $\mathcal{A}$ of rooted trees: $\mathfrak{a}^\bullet=\mathcal{A}$. Indeed, a rooted tree is nothing more than a tree with a distinguished element, its root (see Figure 2.23).

   It is important to note that the distinguished element u of an $F^\bullet$-structure belongs to the underlying set U, whereas the element $*$ of an F'-structure is always outside of the underlying set
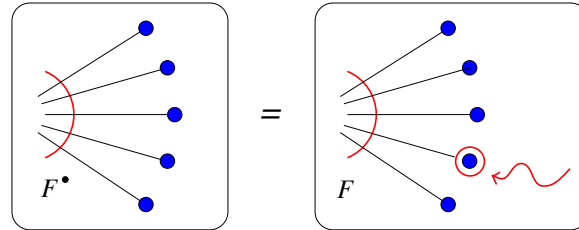
Figure 2.22: Pointing an F-structure.
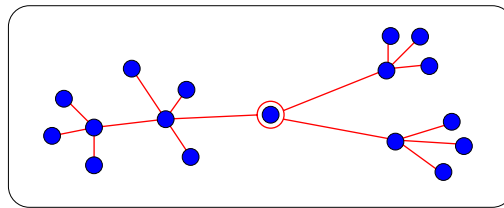
Figure 2.23: A rooted tree.

U. The operations of pointing and derivation are related by the combinatorial equation $F^\bullet = X \cdot F'$, where X denotes the species of singletons. To see this, examine Figure 2.24. The distinguished element (the circled singleton) in the $F^\bullet$-structure on the left is taken aside and is replaced by a $*$. This gives , in a natural fashion an $X \cdot F'$-structure. From the combinatorial equation $F^\bullet = X \cdot F'$,
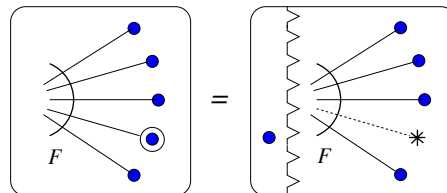
Figure 2.24: Pointing in term of derivation.

we can deduce the main properties of the operation of pointing (see Exercise 2.40), as well as the following proposition concerning the passage to the generating and index series.

Proposition 2.33. *Let F be a species of structures. One has the equalities*

a) $F^\bullet(x) = x \dfrac{d}{dx} F(x),$

b) $\widetilde{F^\bullet}(x) = x \left(\dfrac{\partial}{\partial x_1} Z_F \right)(x, x^2, x^3, \ldots),$

c) $Z_{F^\bullet}(x_1, x_2, x_3, \ldots) = x_1 \left(\dfrac{\partial}{\partial x_1} Z_F \right)(x_1, x_2, x_3, \ldots).$

Example 2.34. In [158], Joyal uses the operation of pointing to determine, in a simple and elegant fashion, the number α_n of trees on a set of n elements (see also [133], 3.3.19). To this end, point twice the species $\mathfrak{a}$ of trees. This gives, by definition, the species $\mathcal{V}$ of **vertebrates**: $\mathcal{V} = \mathfrak{a}^{\bullet\bullet}$. Equivalently, we have $\mathcal{V} = \mathcal{A}^{\bullet}$. Hence, a vertebrate is a **bipointed** tree (or a pointed rooted tree). It possesses in a natural fashion a **vertebral column** (see Figure 2.25), that is, the unique elementary path (refer to the thick lines) from the first distinguished vertex (called the **tail** vertex, labelled by 1) to the second distinguished vertex (called the **head** vertex, labelled by 2). Note that
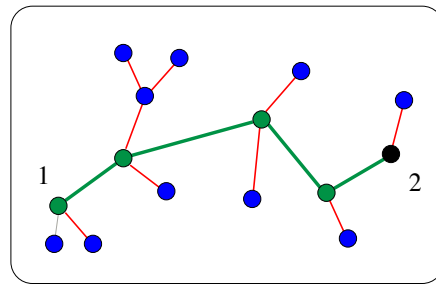
Figure 2.25: A vertebrate.

the tail vertex can coincide with the head vertex. In this case, the vertebrate is called **degenerate**. Denote by ν_n, the number of vertebrates on a set of n elements. We then have $\nu_n = n^2\alpha_n$, since there are n possible choices for the tail vertex and n other (independent) choices for the head vertex. We next calculate ν_n in another fashion. The vertebral column determines in a natural manner a non-empty sequence of disjoint rooted trees (see Figure 2.26). Thus the species $\mathcal{V}$ of vertebrates
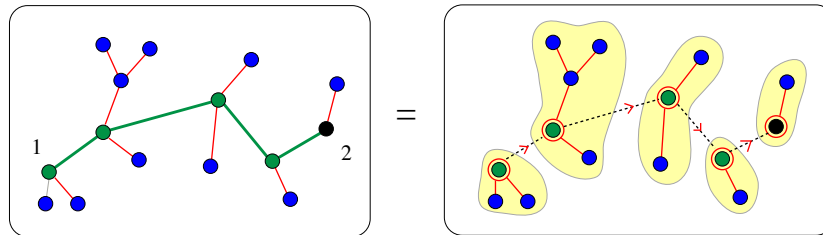
Figure 2.26: A vertebrate as a list of rooted trees.

satisfies the combinatorial equation $\mathcal{V} = L_+(\mathcal{A})$. Replacing in this equality the species L_+ by the equipotent species $\mathcal{S}_+$ of non-empty permutations yields the equipotence $\mathcal{V} \equiv \mathcal{S}_+(\mathcal{A})$. It follows that, on a given set, there are as many vertebrates as (non-empty) permutations of rooted trees and then, as many as (non-empty) endofunctions, since $\mathcal{S}(\mathcal{A}) = \text{End}$. Thus, we have established the equipotence $\mathcal{V} \equiv \text{End}_+$, from which we deduce $\nu_n = n^n$, $n \geq 1$, since $|\text{End}[n]| = n^n$. From $n^2\alpha_n = \nu_n = n^n$, $n \geq 1$, we conclude that $\alpha_n = n^{n-2}$, when $n \geq 1$. This is the classic formula of Cayley for the number of labelled trees. Incidentally, we have also shown the equality $a_n = n^{n-1}$, where a_n denotes the number of rooted trees on n elements, since $a_n = n\,\alpha_n$ (n choices for the

root). For the cycle index series, the combinatorial equation $\mathcal{V} = L_+(\mathcal{A})$ immediately gives

$$Z_{\mathcal{V}}(x_1, x_2, x_3, \ldots) = \frac{Z_{\mathcal{A}}(x_1, x_2, x_3, \ldots)}{1 - Z_{\mathcal{A}}(x_1, x_2, x_3, \ldots)},$$

since $Z_{L_+} = x_1/(1 - x_1)$. It is interesting to note that the equipotence $\mathcal{V} \equiv \mathrm{End}_+$ is not an isomorphism. In fact,

$$
\begin{aligned}
Z_{\mathrm{End}_+} &= Z_{\mathcal{S}_+} \circ Z_{\mathcal{A}} \\
&= \frac{1}{(1 - Z_{\mathcal{A}}(x_1, x_2, x_3, \ldots)) \, (1 - Z_{\mathcal{A}}(x_2, x_4, x_6, \ldots)) \ldots} - 1,
\end{aligned}
$$

so that $Z_{\mathcal{V}} \neq Z_{\mathrm{End}_+}$, which implies $\mathcal{V} \neq \mathrm{End}_+$.

Consider now a species F possessing a notion of connected components, that is to say (as we have seen in Section 2.2) of the form $F = E(F^c)$, where F^c is the species of **connected** F-structures. The operation of pointing permits a straightforward calculation of the average number, $\kappa_n(F)$, of connected components of a random F-structure on n elements, by the formula

$$\kappa_n(F) = \frac{|(F^c \cdot F)[n]|}{|F[n]|}, \quad n \geq 0, \tag{2.22}$$

if $|F[n]| \neq 0$. Indeed, as an $E^{\bullet}(F^c)$-structure can be identified with an F-structure in which a connected component has been distinguished, it suffices to substitute the species F^c in the species $E^{\bullet} = X \cdot E$ to obtain (2.22). Applying formula (2.22) to the species Par, $\mathcal{S}$ and $\mathcal{G}$, gives:

– The average number of classes of a random partition on n elements is, in virtue of equations in Example 2.26,

$$\kappa_n(\mathrm{Par}) = \frac{B_{n+1}}{B_n} - 1, \tag{2.23}$$

where B_n denotes the number of partitions of a set having n elements (Bell number).

– The average number of cycles of a random permutation on n elements is

$$\kappa_n(\mathcal{S}) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \sim \log(n). \tag{2.24}$$

– The average number of connected components of a random simple graph on n vertices is

$$\kappa_n(\mathcal{G}) = 2^{-\binom{n}{2}} \sum_{i=1}^{n} \binom{n}{i} 2^{\binom{n-i}{2}} |\mathcal{G}^c[i]|, \tag{2.25}$$

where $|\mathcal{G}^c[i]|$ is the number of connected graphs on i elements.

2.3.2   Cartesian product of species of structures

The Cartesian product corresponds at a combinatorial level to the coefficient-wise product of exponential generating series, called **Hadamard product** and denoted by $\times$:

$$\left(\sum_{n\geq 0} f_n \frac{x^n}{n!}\right) \times \left(\sum_{n\geq 0} g_n \frac{x^n}{n!}\right) = \sum_{n\geq 0} f_n\, g_n \frac{x^n}{n!}. \qquad\qquad vingtun$$

Definition 2.35. Let F and G be two species of structures. The species $F\times G$, called the **cartesian product** of F and G, is defined as follows: an $(F\times G)$-structure on a finite set U is a pair $s = (f, g)$, where

  i) f is an F-structure on U,

  ii) g is a G-structure on U.

In other words, for all finite sets U, one has $(F \times G)[U] = F[U] \times G[U]$ (Cartesian product). The transport along a bijection $\sigma : U \longrightarrow V$ is carried out by setting $(F\times G)[\sigma](s) = (F[\sigma](f), G[\sigma](g))$, for any $(F \times G)$-structure $s = (f, g)$ on U. An arbitrary $(F \times G)$-structure can be represented by a diagram of the type of Figure 2.27. The labelled enumeration of $F \times G$-structures satisfies

$$|(F \times G)[n]| = |F[n]|{\cdot}|G[n]|, \quad n \geq 0.$$
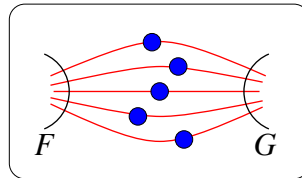
Figure 2.27: A typical structure of species $F \times G$.

Remark 2.36. We underline that $F \times G$ is different from $F \cdot G$: each of the structures f and g appearing in the formation of an $(F \times G)$-structure on U, has underlying set U (in its entirety). However, for $(F \cdot G)$-structures (f, g) on U, the underlying sets U_1 and U_2 of f and g are disjoint (and $U_1 \cup U_2 = U$). The product $F \times G$ is sometimes called the **superposition** of F and G since an $(F \times G)$-structure on U is obtained by superposing an F-structure on U and a G-structure on U.

Example 2.37. Consider the species $\mathcal{C}$ of oriented cycles and the species $\mathfrak{a}$ of trees. Figure 2.28 illustrates the difference between an $(\mathfrak{a} \times \mathcal{C})$-structure and an $(\mathfrak{a} \cdot \mathcal{C})$-structure on a set of seven elements.
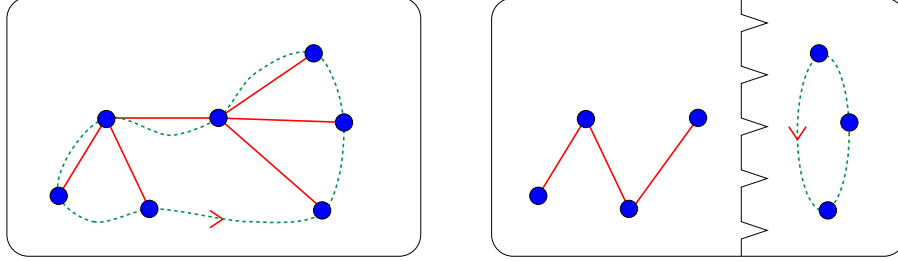
Figure 2.28: An $(\mathfrak{a} \times \mathcal{C})$-structure versus an $(\mathfrak{a} \cdot \mathcal{C})$-structure.

In order to describe the compatibility of the Cartesian product with passage to series, it is necessary to first define the **Hadamard product** $f \times g$ of two index series

$$f(\mathbf{x}) = \sum f_{\mathbf{n}} \frac{\mathbf{x^n}}{\operatorname{aut}(\mathbf{n})}, \qquad g(\mathbf{x}) = \sum g_{\mathbf{n}} \frac{\mathbf{x^n}}{\operatorname{aut}(\mathbf{n})},$$

where $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, and $\mathbf{n} = (n_1, n_2, n_3, \ldots)$. These series are multiplied coefficient-wise:

$$(f \times g)(\mathbf{x}) = \sum f_{\mathbf{n}} \, g_{\mathbf{n}} \frac{\mathbf{x^n}}{\operatorname{aut}(\mathbf{n})}.$$

We then have the following result, whose proof is left as an exercise.

Proposition 2.38. *Let F and G be two species of structures. Then the series associated to the species $F \times G$ satisfy the equalities*

a) $(F \times G)(x) = F(x) \times G(x),$

b) $(\widetilde{F \times G})(x) = (Z_F \times Z_G)(x, x^2, x^3, \ldots),$

c) $Z_{F \times G}(x_1, x_2, x_3, \ldots) = Z_F(x_1, x_2, x_3, \ldots) \times Z_G(x_1, x_2, x_3, \ldots).$

Example 2.39. Consider the Cartesian product $\mathcal{C} \times \wp$ of the species $\mathcal{C}$ of oriented cycles by the species $\wp$ of subsets (of sets). A $(\mathcal{C} \times \wp)$-structure on a set U is then an oriented cycle on U on which one has superimposed a subset V of U (see Figure 2.29). In other words, a $(\mathcal{C} \times \wp)$-structure is an oriented cycle in which certain elements have been distinguished (the circled points in Figure 2.29). We now compute explicitly the series associated to this species. Since $|\mathcal{C}[n]| = (n-1)!$ if $n \geq 1$ and $|\wp[n]| = 2^n$, we obtain

$$|(\mathcal{C} \times \wp)[n]| = \begin{cases} (n-1)! \, 2^n, & \text{if } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We conclude that

$$\begin{aligned} (\mathcal{C} \times \wp)(x) &= \sum_{n \geq 1} (n-1)! \frac{(2x)^n}{n!} \\ &= \log\left(\frac{1}{1-2x}\right). \end{aligned}$$
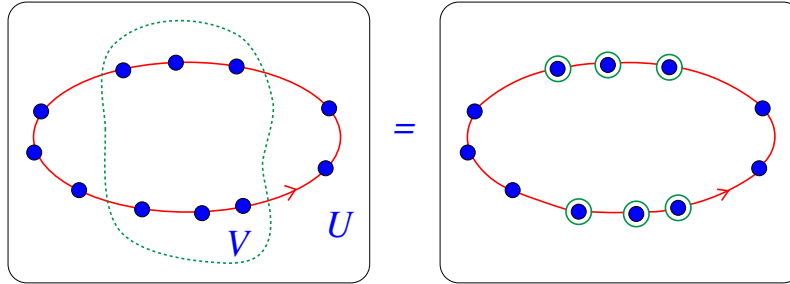
Figure 2.29: Oriented cycle with distinguished points.

A similar calculation, using the fact that fix $\wp[n_1, n_2, \ldots] = 2^{n_1+n_2+\cdots}$ and formula (2.20) for $Z_\mathcal{C}$, yields

$$Z_{\mathcal{C}\times\wp}(x_1, x_2, x_3, \ldots) = \sum_{k\geq1} \frac{\phi(k)}{k} \log\left(\frac{1}{1-2x_k}\right).$$

It follows that the type generating series of $(C\times\wp)$-structures is of the form

$$\widetilde{(\mathcal{C}\times\wp)}(x) = \sum_{k\geq1} \frac{\phi(k)}{k} \log\left(\frac{1}{1-2x^k}\right).$$

We finally deduce that the number of unlabeled $(\mathcal{C}\times\wp)$-structures on a set having n elements is:

$$[x^n]\,\widetilde{(\mathcal{C}\times\wp)}(x) = \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}}.$$

The species E of sets is the neutral element for the Cartesian product, i.e., for any species F, one has

$$E \times F = F \times E = F.$$

Indeed, superimposing an F-structure on a set structure reduces to simply considering the F-structure. By restricting to the cardinality n, it is easy to verify that

$$E_n \times F = F \times E_n = F_n.$$

As the Cartesian product distributes over addition (see Exercise 2.44), we recover the canonical decomposition of a species

$$\begin{aligned} F &= F \times E \\ &= F \times (E_0 + E_1 + E_2 + \ldots + E_n + \ldots) \\ &= F_0 + F_1 + F_2 + \ldots + F_n + \ldots . \end{aligned}$$

For the Cartesian product, the operation of pointing can be distributed on one or the other factor:

$$(F \times G)^\bullet = F^\bullet \times G = F \times G^\bullet,$$
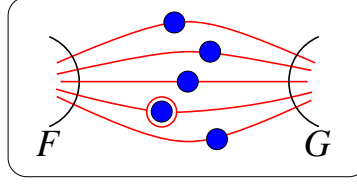
Figure 2.30: Pointing a $F \times G$-structure.

as can be seen from Figure 2.30. In particular, by taking $G = E$ (the species of sets), we obtain the equalities

$$F^\bullet = (F \times E)^\bullet = F \times E^\bullet = F \times (X \cdot E).$$

Hence, the operation of pointing can be expressed in terms of the Cartesian product and ordinary product.

It is interesting to note that the law of simplification (by a non-zero factor) is not valid for the Cartesian product of species. For example, $L \times L = \mathcal{S} \times L$, but $L \neq \mathcal{S}$.

To end this section, let us now consider the series associated with the species $\mathfrak{a} \times \mathcal{C}$ mentioned in Example 6. As $|\mathfrak{a}[n]| = n^{n-2}$ and $|\mathcal{C}[n]| = (n-1)!$ if $n \geq 1$, we obtain the generating series

$$(\mathfrak{a} \times \mathcal{C})(x) = \sum_{n \geq 1} n^{n-3} x^n.$$

For the index series $Z_{\mathfrak{a} \times \mathcal{C}}$, the situation is more delicate since we do not yet know an expression for $Z_{\mathfrak{a}}$. Nevertheless, we can already affirm that many coefficients of $Z_{\mathfrak{a} \times \mathcal{C}}$ are zero. Indeed, we have seen the series $Z_{\mathcal{C}}$ in the form

$$Z_{\mathcal{C}} = \sum_{k \geq 1} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - x_k} \right) = \sum_{k,m \geq 1} \frac{\phi(k) x_k^m}{k\, m}.$$

Hence, only monomials of the form x_k^m enter into play in the expression of $Z_{\mathcal{C}}$. It then follows that $\mathrm{fix}\,\mathcal{C}[n_1, n_2, \ldots] = 0$, except if $(n_1, n_2, \ldots) = (0, 0, \ldots, i, 0, \ldots)$, $i = n_k$, $k \in \mathbb{N}$. We can thus assert (without knowing $Z_{\mathfrak{a}}$) that $Z_{\mathfrak{a} \times \mathcal{C}}$ is of the form

$$Z_{\mathfrak{a} \times \mathcal{C}} = w_1(x_1) + w_2(x_2) + \ldots + w_k(x_k) + \ldots \, ,$$

for certain formal power series $w_k(x) \in \mathbb{Q}[\![x]\!]$. In fact, these series are identically zero, except $w_1(x)$ and $w_2(x)$. Thus, $Z_{\mathfrak{a} \times \mathcal{C}}$ only depends upon x_1 and x_2:

$$Z_{\mathfrak{a} \times \mathcal{C}} = w_1(x_1) + w_2(x_2).$$

The series $Z_{\mathfrak{a}}$, $w_1(x_1)$ and $w_2(x_2)$ are explicitly calculated in Chapter 4 of [22].

2.4 Functorial composition

Definition 2.40. Let F and G be two species of structures. The species $F \square G$ (also denoted by $F[G]$), called the **functorial composite** of F and G, is defined as follows: an $(F \square G)$-structure on U is an F-structure placed on the set $G[U]$ of all the G-structures on U. In other words, for any finite set U, $(F \square G)[U] = F[G[U]]$. The transport along a bijection $\sigma : U \longrightarrow V$ is carried out by setting

$$(F \square G)[\sigma] = F[G[\sigma]] \tag{2.26}$$

(i.e., F-transport along the bijection $G[\sigma]$). As a functor, the species $F \square G$ is the composite of the functors F and G, hence its name. The notation $\square$ is used in this text to avoid ambiguity with the substitution (partitional composition) of species of structures. A generic $(F \square G)$-structure on U can be represented by a diagram of the type in Figure 2.31, where, on the right hand side, it is understood that the totality of G-structures on U appear. The labelled enumeration of the $F \square G$-structures satisfies $|(F \square G)[n]| = |F[g_n]|$, with $g_n = |G[n]|$, which corresponds to an operation on the exponential formal power series, also denoted by $\square$:

$$\left(\sum_{n \geq 0} f_n \frac{x^n}{n!} \right) \square \left(\sum_{n \geq 0} g_n \frac{x^n}{n!} \right) = \sum_{n \geq 0} f_{g_n} \frac{x^n}{n!},$$

under the hypothesis that $g_n \in \mathbb{N}$, for any $n \in \mathbb{N}$.
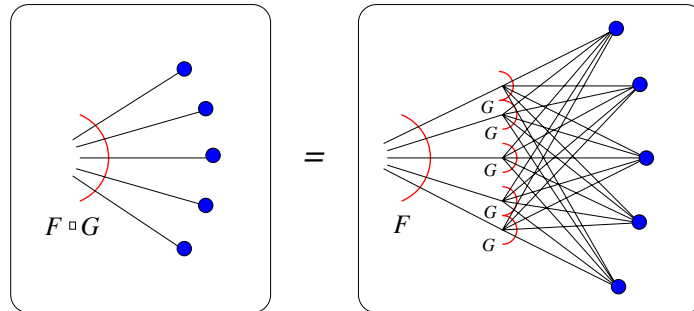
Figure 2.31: A typical $F \square G$-structure.

Example 2.41. Using functorial composition of species, we can express a variety of graphs classes (simple, directed, with or without loops, etc.) in terms of simple species of structures. For example, the species $\mathcal{G}$ of all simple graphs (without loops) can be expressed as $\mathcal{G} = \wp \square \wp^{[2]}$, where $\wp$ denotes the species of subsets and $\wp^{[2]}$, that of subsets with two elements. Indeed a $\wp^{[2]}$-structure on a set amounts to considering a pair of elements, joined by a segment in Figure 2.32 a). Such a pair of elements is called an **edge**. Moreover, a graph on a set U is nothing else but a selection among all possible edges. There is then a $\wp$-structure on the set of all $\wp^{[2]}$-structures on U. The reader can verify without difficulty that the transport of structures is as described in (2.26). Observe that,

since $\wp = E \cdot E$ and $\wp^{[2]} = E_2 {\cdot} E$, the species of graphs can also be expressed in the form

$$\mathcal{G} = (E \cdot E) \,\square\, (E_2 \cdot E), \tag{2.27}$$

which only uses the species of sets, the product, the functorial composition, and the restriction to the cardinality 2.
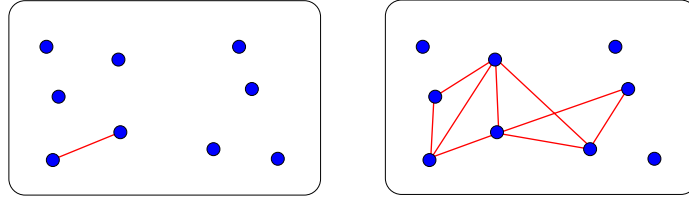
Figure 2.32: a) A typical $\wp^{[2]}$-structure.   b) A typical $\wp \,\square\, \wp^{[2]}$-structure.

The operation $\square$ is clearly associative but is not commutative (not even up to isomorphism). For example, the species $\wp^{[2]} \,\square\, \wp$ is identified with the species of **pairs of subsets**. It is not isomorphic to the species $\mathcal{G} = \wp \,\square\, \wp^{[2]}$ of graphs. The species $E^{\bullet}$ of pointed sets is the **neutral element** for the operation $\square$ (see Exercise 2.53), i.e., for any species F, $F \,\square\, E^{\bullet} = E^{\bullet} \,\square\, F = F$. In order to describe the behavior of the composition of species with respect to power series, it is convenient to first define a corresponding operation $Z_F \,\square\, Z_G$ in the context of cycle index series. It clearly follows from the definition of the transport of $(F \,\square\, G)$-structures that

$$
\begin{aligned}
\text{fix}\,(F \,\square\, G)[\sigma] &= \text{fix}\, F[G[\sigma]] \\
&= \text{fix}\, F[(G[\sigma])_1, (G[\sigma])_2, \ldots],
\end{aligned}
$$

for any permutation σ where $[(G[\sigma])_1, (G[\sigma])_2, \ldots]$ denotes the cycle type of the permutation $G[\sigma]$, as defined in Section 1.2. The following proposition shows that the index series Z_G completely determines this cycle type.

Proposition 2.42. *Let G be a species of structures, $\sigma \in \mathcal{S}_n$, and $k \geq 1$. Then the number of cycles of length k in $G[\sigma]$ is given by*

$$(G[\sigma])_k = \frac{1}{k} \sum_{d \mid k} \mu(\frac{k}{d})\, \text{fix}\, G[\sigma^d], \tag{2.28}$$

where μ denotes the Möbius function for positive integers.

Proof. For any permutation β, and any $k \geq 1$,

$$\text{fix}\, \beta^k = \sum_{d \mid k} d\, \beta_d. \tag{2.29}$$

Indeed, an element is left fixed by the permutation β^k if and only if it is found in a cycle of β of length d, where d divides k. By applying Möbius inversion to (2.29), we deduce

$$\beta_k = \frac{1}{k} \sum_{d \mid k} \mu(k/d) \text{ fix } \beta^d. \tag{2.30}$$

If $\beta = G[\sigma]$, then by functoriality $\beta^d = G[\sigma^d]$, giving the result. $\qquad\square$

In other words, the knowledge of the coefficients of the cycle index series Z_G make possible the calculation of all the $(G[\sigma])_k$. The following definition is thus legitimate.

Definition 2.43. The **composite** $Z_F \square Z_G$ is defined by the formula

$$Z_F \square Z_G = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \text{fix } F[(G[\sigma])_1, (G[\sigma])_2, \ldots] \; x_1^{\sigma_1} x_2^{\sigma_2} \ldots,$$

where $(\sigma_1, \sigma_2, \ldots)$ denotes the cycle type of a permutation $\sigma \in \mathcal{S}_n$, $n = 0, 1, 2, \ldots$.

Immediately implied is the following result.

Proposition 2.44. *Let F and G be two species of structures. Then the series associated to the species $F \square G$ satisfy the equalities*

a) $(F \square G)(x) = F(x) \square G(x),$

b) $\widetilde{(F \square G)}(x) = (Z_F \square Z_G)(x, x^2, x^3, \ldots),$

c) $Z_{F \square G}(x_1, x_2, x_3, \ldots) = Z_F(x_1, x_2, x_3, \ldots) \square Z_G(x_1, x_2, x_3, \ldots).$

Example 2.45. A particularly interesting case is $F = \wp$, the species of subsets. Many varieties of graphs and relations can be described as the composite $\wp \square G$ of $\wp$ and of a given species G. Here are some examples:

  – Simple graphs: $\mathcal{G} = \wp \square \wp^{[2]}$;

  – Directed graphs: $\mathcal{D} = \wp \square (E^\bullet \times E^\bullet)$;

  – m-ary relations: $\text{Rel}^{[m]} = \wp \square (E^\bullet)^{\times m}$, where $(E^\bullet)^{\times m} = \underbrace{E^\bullet \times \cdots \times E^\bullet}_{(m \text{ factors})}$.

These species have the respective generating series

$$\mathcal{G}(x) = \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}, \qquad \mathcal{D}(x) = \sum_{n \geq 0} 2^{n^2} \frac{x^n}{n!}, \qquad \text{Rel}^{[m]}(x) = \sum_{n \geq 0} 2^{n^m} \frac{x^n}{n!}.$$

We are now going to calculate the cycle index series of such species using the fact that for any permutation β,

$$\mathrm{fix}\,\wp[\beta] = 2^{\sum_{k\geq 1}\beta_k},$$

to obtain the following proposition.

Proposition 2.46. *For any species of structures G and any permutation σ,*

$$\mathrm{fix}\,(\wp\square G)[\sigma] = 2^{\sum_{k\geq 1}(G[\sigma])_k}.$$

Example 2.47. Thus, the cycle index series of a species of the form $\wp\square G$ simply depends on the numbers $(G[\sigma])_k$. For the three preceding examples, these numbers are given by the following expressions:

a) $\qquad (\wp^{[2]}[\sigma])_k = \dfrac{1}{2}\sum_{[i,j]=k}(i,j)\sigma_i\sigma_j + \sigma_{2k} - \sigma_k + \dfrac{1}{2}(k \bmod 2)\sigma_k,$ (2.31)

b) $\qquad ((E^\bullet \times E^\bullet)[\sigma])_k = \sum_{[i,j]=k}(i,j)\sigma_i\sigma_j,$ (2.32)

c) $\qquad ((E^\bullet)^{\times m}[\sigma])_k = \sum_{[j_1,\ldots,j_m]=k}\dfrac{j_1\cdots j_m}{k}\,\sigma_{j_1}\ldots\sigma_{j_m},$ (2.33)

where σ is of cycle type $(\sigma_1,\sigma_2,\sigma_3,\ldots)$, and $[j_1,\ldots,j_m]$ denotes the least common multiple of $j_1,\ldots,j_m$ and $(j_1,\ldots,j_m)$, the greatest common divisor. These formulas can be established directly by simple counting arguments. For example, for (2.32), recall that $(E^\bullet \times E^\bullet)[U] = U \times U$. Then a pair of elements $(a,b) \in U \times U$, where a belongs to a cycle of length i of σ and b to a cycle of length j, generates a cycle $(a,b) \to (\sigma(a),\sigma(b)) \to (\sigma^2(a),\sigma^2(b)) \to \ldots$ of length $[i,j]$. Each of the $\sigma_i\sigma_j$ cycles induces $(i,j) = ij/[i,j]$ such cycles, whence (2.32). Formula (2.33) is proven in the same manner, by induction on m. A direct combinatorial proof of (2.31) is proposed in Exercise 2.48. We present in what follows a more algebraic proof, based on relation (2.28). Observe at first that for any permutation σ,

$$\mathrm{fix}\,\wp^{[2]}[\sigma] = \binom{\sigma_1}{2} + \sigma_2$$ (2.34)

since a pair of elements left fixed by σ either consists of a pair of fixed points of σ or of a cycle of length 2 of σ. We obtain, in virtue of (2.28), (2.30), and (2.34) and the following lemma,

$$
\begin{aligned}
(\wp^{[2]}[\sigma])_k &= \frac{1}{k}\sum_{d|k}\mu(k/d)\mathrm{fix}\,\wp^{[2]}[\sigma^d] \\
&= \frac{1}{k}\sum_{d|k}\mu(k/d)\frac{1}{2}\big((\sigma^d)_1^2 - (\sigma^d)_1 + 2(\sigma^d)_2\big) \\
&= \frac{1}{2}\big(\sum_{[i,j]=k}(i,j)\sigma_i\sigma_j - \sigma_k + 2\sigma_{2k} - \sigma_k\chi(k \text{ is even})\big),
\end{aligned}
$$

whence formula (2.31).

Lemma 2.48. *For any permutation σ and any $k \geq 1$,*

$$\text{a)} \qquad \frac{1}{k}\sum_{d|k}\mu(\frac{k}{d})(\sigma^d)_1^2 = \sum_{[i,j]=k}(i,j)\sigma_i\sigma_j, \tag{2.35}$$

$$\text{b)} \qquad \frac{1}{k}\sum_{d|k}\mu(\frac{k}{d})(\sigma^d)_2 = \begin{cases} \sigma_{2k} & \text{if } k \text{ is odd,} \\ \sigma_{2k} - \frac{1}{2}\sigma_k, & \text{otherwise.} \end{cases} \tag{2.36}$$

Proof. To prove (2.35), it suffices to apply Proposition 3 with $G = E^\bullet \times E^\bullet$, i.e., to join the identities (2.28) and (2.32), observing also that for any permutation τ, fix $(E^\bullet \times E^\bullet)[\tau] = \tau_1^2$. For (2.36), we have, by (2.30), $(\sigma^d)_2 = \frac{1}{2}\big((\sigma^{2d})_1 - (\sigma^d)_1\big)$ and

$$\begin{aligned}(\sigma^{2d})_1 &= \sum_{i|2d} i\,\sigma_i \\ &= \sum_{j|d} 2j\,\sigma_{2j} + \sum_{i|d} i\,\sigma_i\,\chi(i \text{ is odd}).\end{aligned}$$

Using the fact that for any function f,

$$\sum_{d|k}\mu(\frac{k}{d})\sum_{i|d}f(i) = f(k),$$

we obtain

$$\frac{1}{k}\sum_{d|k}\mu(\frac{k}{d})(\sigma^d)_2 = \frac{1}{2k}(2k\,\sigma_{2k} + k\,\sigma_k\,\chi(k \text{ is odd}) - k\sigma_k),$$

whence (2.36). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Combining Propositions 2.47 with the formulas of example 2.47, we find for the species $\mathcal{G}$ of simple graphs, $\mathcal{D}$ of directed graphs, and $\text{Rel}^{[m]}$ of m-ary relations, that for any permutation σ,

$$\begin{aligned}\text{fix}\,\mathcal{G}[\sigma] &= 2^{\frac{1}{2}\sum_{i,j\geq 1}(i,j)\sigma_i\sigma_j - \frac{1}{2}\sum_{k\geq 1}(k \bmod 2)\sigma_k}, \\ \text{fix}\,\mathcal{D}[\sigma] &= 2^{\sum_{i,j\geq 1}(i,j)\sigma_i\sigma_j}, \\ \text{fix}\,\text{Rel}^{[m]}[\sigma] &= 2^{\sum_{i_1,\dots,i_m\geq 1} i_1\dots i_m\, \sigma_{i_1}\dots\sigma_{i_m}/[i_1,\dots,i_m]}.\end{aligned}$$

These expressions permit the calculation of the cycle index series and the type generating series of these species. For example,

$$\widetilde{\mathcal{G}}(x) = 1 + x + 2x^2 + 4x^3 + 11x^4 + 34x^5 + 156x^6 + 1044x^7 + 12346x^8 + 274668x^9 + \dots$$

2.5   Exercises

Exercises for Section 2.1

Exercise 2.1. Let F, G and H be species of structures.

a) Show, by explicitly describing the isomorphisms, that addition has the following properties:

   i) $(F + G) + H = F + (G + H)$, (associativity)
   ii) $F + G = G + F$, (commutativity)
   iii) $F + 0 = 0 + F = F$. (neutral element)

b) Show that the passage to generating and cycle index series preserves the operation of addition (see the identities in (2.1)).

Exercise 2.2.   a) Let $(F_i)_{i \in I}$ be a summable family of species of structures. Show that the sum $\sum_{i \in I} F_i$ of this family, defined by (2.3) and (2.4), is indeed a species of structures.

b) Let $(h_i(x_1, x_2, \dots))_{i \in I}$ be a family of formal series in the variables $x_1, x_2, \dots$, expressed in the form

$$h_i(x_1, x_2, \dots) = \sum_{n_1, n_2, \dots} h_{i; n_1, n_2, \dots} \frac{x_1^{n_1} x_2^{n_2} \cdots}{c_{n_1, n_2, \dots}}, \quad i \in I,$$

where $c_{n_1, n_2, \dots}$ is a given family of non-zero scalars. By definition, the family

$$(h_i(x_1, x_2, \dots))_{i \in I}$$

is said to be **summable** if, for each multi-index $n_1, n_2, \dots$, one has

$$c_{n_1, n_2, \dots} [x_1^{n_1} x_2^{n_2} \dots] h_i(x_1, x_2, \dots) = h_{i; n_1, n_2, \dots} = 0,$$

except for a finite number of $i \in I$. The sum of the family is the formal series $h(x_1, x_2, \dots)$ whose coefficients are given by the (finite) sums

$$c_{n_1, n_2, \dots} [x_1^{n_1} x_2^{n_2} \dots] h(x_1, x_2, \dots) = \sum_{i \in I} h_{i; n_1, n_2, \dots}.$$

Show that, if $(F_i)_{i \in I}$ is a summable family of species of structures, then the families of formal series

$$(F_i(x))_{i \in I}, \quad \left(\widetilde{F}_i(x) \right)_{i \in I} \quad \text{and} \quad (Z_{F_i}(x_1, x_2, \dots))_{i \in I}$$

are summable and that equalities (2.5) hold.

Exercise 2.3. Let F be a species of structures. The **even part** and **odd part** of F are the species defined by the decompositions $F_{\text{even}} = F_0 + F_2 + F_4 + \dots$ and $F_{\text{odd}} = F_1 + F_3 + F_5 + \dots$. Show that the following equalities are satisfied:

a) $Z_{F_{\text{even}}} = \dfrac{1}{2} \left(Z_F(x_1, x_2, x_3, x_4, \ldots) + Z_F(-x_1, x_2, -x_3, x_4, \ldots) \right),$

b) $Z_{F_{\text{odd}}} = \dfrac{1}{2} \left(Z_F(x_1, x_2, x_3, x_4, \ldots) - Z_F(-x_1, x_2, -x_3, x_4, \ldots) \right).$

In the case where $F = E$, the species of sets, deduce formulas (2.2).

Exercise 2.4.   a) Let F, G and H be species of structures. Describe the isomorphisms establishing the following properties of multiplication:

    i) $(F \cdot G) \cdot H = F \cdot (G \cdot H)$, (associativity)

    ii) $F \cdot G = G \cdot F$, (commutativity)

    iii) $F \cdot 1 = 1 \cdot F = F$, (neutral element)

    iv) $F \cdot 0 = 0 \cdot F = 0$, (absorbing element)

    v) $F \cdot (G + H) = F \cdot G + F \cdot H$. (distributivity)

b) Prove identities (2.7) concerning the series associated with the product $F \cdot G$ of two species of structures.

c) Using the definitions of sum and product of species, prove that, for all integers $n \geq 0$, one has the combinatorial equality

$$\underbrace{F + F + \ldots + F}_{n \text{ times}} = n \cdot F,$$

where, by convention, the n on the right-hand side denotes the species having exactly n structures on the empty set and no structure on other sets.

d) Let $n \geq 0$ be an integer and set $F = n! \cdot E_n$ and $G = X^n$. For which values of n are the species F and G isomorphic?

Exercise 2.5.   a) Show that the species $\wp$, where $\wp[U] = \{B \mid B \subseteq U\}$, is isomorphic to the species $E \cdot E$.

b) Show directly that the number of structures of the species $\wp$ on a set U that are fixed by a permutation of U of cycle type $(n_1, n_2, n_3, \ldots)$ is

$$1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \ldots [x_1^{n_1} \, x_2^{n_2} \, x_3^{n_3} \ldots] Z_\wp(x_1, x_2, x_3, \ldots) = 2^{n_1 + n_2 + n_3 + \cdots},$$

by partitioning the cycles of the permutation according to whether or not they are contained in the considered subset.

Exercise 2.6. Show that, for the species Der of derangements, one has

$$\text{fix Der}\,[n_1, n_2, \ldots] = n_1! \, n_2! \ldots \sum_{\substack{0 \leq i_k \leq n_k \\ k \geq 1}} \frac{(-1)^{i_1 + i_2 + \cdots} \, 1^{n_1 - i_1} 2^{n_2 - i_2} \cdots}{i_1! \; i_2! \; \ldots}.$$

Exercise 2.7. Show that the species $\wp^{[k]}$, where $\wp^{[k]}[U] = \{B \mid B \subseteq U \text{ and } |B| = k\}$, is isomorphic to the species $E_k \cdot E$. Also show that

$$\text{fix } \wp^{[k]}[n_1, n_2, n_3, \ldots] = \sum_{k_1 + 2k_2 + 3k_3 + \ldots = k} \prod_{i \geq 1} \binom{n_i}{k_i}.$$

Exercise 2.8. Let F and G be two species of structures. Show that the family $(F_m \cdot G_n)_{(m,n)}$, with $(m, n) \in \mathbb{N} \times \mathbb{N}$, is summable and that

$$F{\cdot}G = \sum_{(m,n)} F_m{\cdot}G_n.$$

Deduce the canonical decomposition of $F{\cdot}G$. (In this problem, F_m denotes the restriction of the species F to the cardinality m.)

Exercise 2.9. Let $(F_i)_{i \in I}$ be a family of species of structures. By convention, a $\prod_{i \in I} F_i$-structure on a finite set U is a family $(s_i)_{i \in I}$ where for each $i \in I$ one has $s_i \in F_i[U_i]$ for a subset $U_i \subseteq U$, and the subsets U_i are required to be pairwise disjoint and to satisfy $\bigcup_{i \in I} U_i = U$. The family $(F_i)_{i \in I}$ is said to be **multiplicable** if for any finite set U, the set of $\prod_{i \in I} F_i$-structures on U is finite.

a) For a multiplicable family $(F_i)_{i \in I}$ of species, define the species product $\prod_{i \in I} F_i$ (do not forget the transports of structures).

b) Show that a non-empty family of species $(F_i)_{i \in I}$ is multiplicable if and only if there exists a $J \subseteq I$ such that

   i) $I \backslash J$ is finite,

   ii) $i \in J \iff (F_i)_0 = 1$,

   iii) the family of $((F_i)_+)_{i \in J}$ is summable.

c) Show that if $(F_i)_{i \in I}$ is multiplicable and if $J = \{i \in I \mid (F_i)_0 = 1\}$, then $(F_j)_{j \in J}$ is multiplicable and one has the combinatorial equalities

   i) $\displaystyle \prod_{i \in I} F_i = \left(\prod_{i \in I \backslash J} F_i \right) \cdot \left(\prod_{j \in J} F_j \right),$

   ii) $\displaystyle \prod_{j \in J} F_j = 1 + \sum_{j \in J} (F_j)_+ + \sum_{\{j_1, j_2\} \in \wp^{[2]}(J)} (F_{j_1})_+ (F_{j_2})_+ + \ldots.$

Exercise 2.10. Denote by $\mathcal{S}^{<k>}$ the species of permutations having all cycles of length k. Show that the infinite family $(\mathcal{S}^{<k>})_{k \geq 1}$ is multiplicable and that the species $\mathcal{S}$ of permutations satisfies

$$\mathcal{S} = \prod_{k \geq 1} \mathcal{S}^{<k>}.$$

Exercise 2.11. Determine whether the species Bal of ballots is isomorphic to the species

$$\prod_{n\geq 1} \left(1 + E_n + E_n^2 + E_n^3 + \ldots\right).$$

Exercise 2.12. Prove formulas (2.10) for the series $\mathrm{Bal}(x)$, $\widetilde{\mathrm{Bal}}(x)$ and $Z_{\mathrm{Bal}}(x_1, x_2, \ldots)$.

    Hint: First establish formulas (2.9).

Exercise 2.13. For the species L of linear orderings, prove the combinatorial equalities

$$L = 1 + X\,L = \sum_{k\geq 0} X^k = \prod_{i\geq 0} \left(1 + X^{2^i}\right).$$

Exercises of Section 2.2

Exercise 2.14. Let F, G, H and K be species of structures, with $G[\emptyset] = \emptyset = H[\emptyset]$.

a) Verify that the partitional composition $F \circ G$, defined by formulas (2.11) and (2.12), is a species of structures.

b) Show, by an explicit description of the isomorphisms, that the partitional composition has the following properties

   i) $(F \circ G) \circ H = F \circ (G \circ H)$, (associativity)

   ii) $F \circ X = X \circ F = F$, (neutral element)

   iii) $(F + K) \circ G = F \circ G + K \circ G$, (distributivity)

   iv) $(F \cdot K) \circ G = (F \circ G) \cdot (K \circ G)$, (distributivity)

   v) $F_0 = F \circ 0$ and $F[\emptyset] = \emptyset$ if and only if $F(0) = 0$.

c) Show, by enumerating the $F \circ G$-structures, that one indeed has $(F \circ G)(x) = F(G(x))$.

Exercise 2.15.   a) Let k be a fixed integer. Verify that the species $\mathcal{S}^{[k]}$ of permutations having k cycles and the species $\mathrm{Par}^{[k]}$ of partitions having k blocks satisfy the combinatorial equations

$$\mathcal{S}^{[k]} = E_k \circ \mathcal{C}, \qquad \mathrm{Par}^{[k]} = E_k \circ E_+,$$

where E_k is the species of sets of cardinality k and $\mathcal{C}$ is that of oriented cycles (cyclic permutations).

b) Let $c(n, k)$ be the number of permutations of a set of cardinality n having k cycles and $S(n, k)$ be the number of partitions of a set of cardinality n having k blocks. The numbers $s(n, k) := (-1)^{n-k} c(n, k)$ and $S(n, k)$ are called, respectively, **Stirling numbers of the first and second kind**. Deduce from a) the following identities:

i) $\displaystyle\sum_{n\geq k} c(n,k)\frac{x^n}{n!} = \frac{(\log 1/(1-x))^k}{k!}$,

ii) $\displaystyle\sum_{n\geq k} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$.

c) Show that for $n \geq 0$, $k \geq 1$,

    i) $S(n+1,k) = S(n,k-1) + k\,S(n,k)$,

    ii) $c(n+1,k) = c(n,k-1) + n\,c(n,k)$.

d) Show that for $n \geq 0$,

    i) $\displaystyle\sum_{k=0}^{n} s(n,k)\,x^k = x_{<n>} := x\,(x-1)\cdots(x-n+1),)$,

    ii) $\displaystyle\sum_{k=0}^{n} S(n,k)\,x_{<k>} = x^n$.

Exercise 2.16. Show that the generating series of unlabeled permutations is not the composite of the type generating series of the species of sets with that of cycles (i.e., $\widetilde{\mathcal{S}}(x) \neq \widetilde{E}(\widetilde{\mathcal{C}}(x))$).

Exercise 2.17. Show that formula (2.13) b), namely

$$\widetilde{(F\circ G)}(x) = Z_F(\widetilde{G}(x), \widetilde{G}(x^2), \widetilde{G}(x^3), \ldots),$$

is a consequence of formula (2.13) c).

Exercise 2.18.   a) Verify the formulas in example 2.17 relating the series $\mathrm{End}(x)$, $\widetilde{\mathrm{End}}(x)$, Z_{End} to the series $\mathcal{A}(x)$, $\widetilde{\mathcal{A}}(x)$ and $Z_{\mathcal{A}}$.

b) Show that the formulas (2.15) implicitly determine the series $\mathcal{A}(x)$, $\widetilde{\mathcal{A}}(x)$ and $Z_{\mathcal{A}}$.

Exercise 2.19.   a) Starting from combinatorial equation (2.16), reprove the explicit formulas (2.10) for the series $\mathrm{Bal}(x)$, $\widetilde{\mathrm{Bal}}(x)$ and Z_{Bal}.

b) Starting from combinatorial equation (2.17), establish formulas (2.18).

c) Establish identity (2.19) a), namely

$$\prod_{k\geq 1} \frac{1}{1-x^k} = \exp\sum_{n\geq 1} \frac{1}{n}\frac{x^n}{1-x^n},$$

starting from the combinatorial equation $\mathcal{S} = E(\mathcal{C})$.

Exercise 2.20. For two formal power series $a = a(x_1, x_2, x_3, \ldots)$ and $b = b(x_1, x_2, x_3, \ldots)$ show that with notational convention (2.14),

$$b = \sum_{k \geq 1} \frac{1}{k} a_k \qquad \Longleftrightarrow \qquad a = \sum_{k \geq 1} \frac{\mu(k)}{k} b_k.$$

Hint: Use the following classic property of the Möbius function μ:

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 2.21.   a) Taking the logarithm of the equality (2.19), b) and using Exercise 2.20, prove the explicit formula (2.20), namely

$$Z_{\mathcal{C}}(x_1, x_2, x_3, \ldots) = \sum_{k \geq 1} \frac{\phi(k)}{k} \log \frac{1}{1 - x_k}.$$

Hint: Use the following formula for the Euler ϕ-function: $\phi(n) = \sum_{d|n} d\, \mu(n/d)$.

b) Deduce from a) the remarkable relation

$$\frac{x}{1 - x} = \sum_{k \geq 1} \frac{\phi(k)}{k} \log \frac{1}{1 - x^k}.$$

Exercise 2.22. Show that if two species of structures F and F^c are related by the combinatorial equation $F = E(F^c)$, i.e., F^c is the species of connected F-structures, then

a) $F^c(x) = \log F(x)$,

b) $\widetilde{F^c}(x) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log \widetilde{F}(x^k)$,

c) $Z_{F^c}(x_1, x_2, \ldots) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log Z_F(x_k, x_{2k}, \ldots)$.

Exercise 2.23. Consider the species Oct of octopuses, introduced in Exercise 1.7.

a) Show that $\mathrm{Oct} = \mathcal{C}(L_+)$.

b) Show, by calculus, that $\mathrm{Oct}(x) = \mathcal{C}(2x) - \mathcal{C}(x)$.

c) This formula suggests the combinatorial equation $\mathrm{Oct}(X) + \mathcal{C}(X) = \mathcal{C}(2X)$. Prove the validity of this equation.

d) Deduce the formulas

   i) $\mathrm{Oct}(x) = \sum_{n \geq 1} (2^n - 1) x^n$,

ii) $Z_{\text{Oct}}(x_1, x_2, x_3, \ldots) = \sum_{n \geq 1} \frac{\phi(n)}{n} \log\left(\frac{1-x_n}{1-2x_n}\right).$

Exercise 2.24.   a) Show that the number of octopuses on n vertices ($n \geq 1$), where each tentacle is of odd length, is given by $(n-1)! \left(\mathcal{L}_n + (-1)^{n+1} - 1\right)$, where

$$\mathcal{L}_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

designates the nth **Lucas number**. **Note:** the Lucas numbers satisfy the recurrence

$$\mathcal{L}_0 = 2, \quad \mathcal{L}_1 = 1, \quad \mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2}, \text{ for } n \geq 2.$$

b) Compute the cycle index series of the species of octopuses with odd tentacles.

Exercise 2.25.   a) Let F and G be two species of structures, with $G(0) = 0$. Show that the family $(F(G_n))_{n \geq 1}$ is summable, where G_n is the species G restricted to the cardinality n. Also give a description of the $\sum_{n \geq 1} F(G_n)$-structures.

b) The combinatorial equation $\text{Oct}_{\text{reg}}(X) = \sum_{n \geq 1} C(X^n)$ defines the species Oct_{reg} of **regular octopuses**. Justify this terminology by giving a graphical example of a Oct_{reg}-structure and establish the following formulas for the series associated to the species Oct_{reg}:

i) $\text{Oct}_{\text{reg}}(x) = \sum_{n \geq 1} \left(\sum_{d|n} d\right) \frac{x^n}{n},$

ii) $\widetilde{\text{Oct}_{\text{reg}}}(x) = \sum_{n \geq 1} \tau(n)\, x^n$, where $\tau(n) = $ number of divisors of n,

iii) $Z_{\text{Oct}_{\text{reg}}}(x_1, x_2, \ldots) = -\sum_{k,n \geq 1} \frac{\phi(k)}{k} \log\left(1 - x_k^n\right).$

c) Show that the species Cha of chains, introduced in Exercise 1.15, can be written in the form

$$\text{Cha} = (1 + X)\left(1 + \sum_{n \geq 1} E_2(X^n)\right),$$

where E_2 designates the species of sets of cardinality two. Deduce the expressions for the various series associated to the species Cha.

Exercise 2.26. Let F and G be two species of structures, with $F(0) = 1$. Show that the family $(F(G_n))_{n \geq 1}$ is multiplicable. Also give a description of the $\prod_{n \geq 1} F(G_n)$-structures.

Exercise 2.27. Let $G = L_+$ be the species of non-empty linear orderings. Show that the iterations $G^{\langle n \rangle}$ of G are given by the combinatorial equations $G^{\langle n \rangle} = X\,L(n\,X)$, $n = 0, 1, 2, \ldots$.

Exercise 2.28. A partially ordered set is called **reduced** if it satisfies the following condition: for all x and y,

$$\left. \begin{array}{l} \{z|z \leq x\} \cup \{y\} = \{z|z \leq y\} \\ \{z|z \geq x\} = \{z|z \geq y\} \cup \{x\} \end{array} \right\} \quad \Rightarrow \quad x = y.$$

Denote by Red the species of reduced partial orderings.

a) Establish the combinatorial equation $\text{Ord} = \text{Red}(L_+)$.

b) Deduce the following relations:

   i) $\text{Red}(x) = \text{Ord}\left(\frac{x}{1+x}\right)$,

   ii) $\widetilde{\text{Red}}(x) = Z_{\text{Ord}}\left(\frac{x}{1+x}, \frac{x^2}{1+x^2}, \ldots\right)$,

   iii) $Z_{\text{Red}}(x_1, x_2, x_3, \ldots) = Z_{\text{Ord}}\left(\frac{x_1}{1+x_1}, \frac{x_2}{1+x_2}, \ldots\right)$.

Exercise 2.29. It is always possible to choose canonically a supplementary element $*_U$ outside of each finite set U by taking $*_U = U$ (i.e., the supplementary "element" is the set U itself!). In this case, we say that U^+ is the **set successor** of U and write $U^+ = U \cup \{U\}$. We have as well $*_U \notin U$, since according to classical set theory, one always has $U \notin U$ (Foundation Axiom). Show that with this choice for $*_U$, the operation of differentiation is well-defined and that indeed $F'(x) = \frac{d}{dx}F(x)$.

Exercise 2.30.   a) Verify the equalities of Proposition 2.25 in the cases: $F = \mathcal{C}$, E and L.

b) Prove formula c) of Proposition 2.25.

Exercise 2.31. Recall that the species Bal of ballots and the species Oct of octopuses are characterized by the combinatorial equations $\text{Bal} = L(E_+)$ and $\text{Oct} = \mathcal{C}(L_+)$. Show, by graphical arguments, that the derivatives of these species satisfy

a) $\text{Bal}' = \text{Bal}^2 \cdot E$,

b) $\text{Oct}' = L(X) \cdot L(2X)$.

Exercise 2.32. Establish rigorously, by giving the explicit isomorphisms, the following rules of combinatorial differential calculus.

a) $(F + G)' = F' + G'$, (additivity)

b) $(F \cdot G)' = F' \cdot G + F \cdot G'$, (product rule)

c) $(F \circ G)' = (F' \circ G) \cdot G'$. (chain rule)

Exercise 2.33. In a purely formal fashion (i.e., by using rules of combinatorial differential calculus, associativity, etc.), establish formulas a) and b) of problem 18 starting from the combinatorial equations $\text{Bal} = L(E_+)$ and $\text{Oct} = \mathcal{C}(L_+)$.

Exercise 2.34. Let $\mathcal{A} = X\,E(\mathcal{A})$, be the species of rooted trees, $\mathcal{F} = E(\mathcal{A})$, that of rooted forests and $\mathfrak{a}$, that of trees. Establish, by combinatorial calculus as well as by a graphical argument, the following combinatorial equations:

a) $\mathcal{A}' = \mathcal{F} \cdot L(\mathcal{A})$,

b) $\mathfrak{a}'' = \mathcal{F} \cdot A'$,

c) $A'' = (A')^2 + (A')^2 L(A)$.

Exercise 2.35. Calculate the successive derivatives of

a) the species $\wp = E^2$ of subsets of a set,

b) the species L of linear orderings.

Exercise 2.36. Let $D = \frac{d}{dx}$ be the differentiation operator with respect to the variable x. In the classic theory of ordinary differential equations, the following identity is very useful: $(D - c)f(x) = e^{cx} D e^{-cx} f(x)$, where c is any constant and $f(x)$ any differentiable function or a formal power series. One also has the more general identity

$$(D + h'(x))f(x) = e^{-h(x)} D\, e^{h(x)} f(x),$$

which can be rewritten in the form

$$e^{h(x)}(f'(x) + h'(x)f(x)) = (e^{h(x)} f(x))'.$$

Establish, by a geometric argument, the following corresponding combinatorial identity, namely $E(H) \cdot (F' + H' \cdot F) = (E(H) \cdot F)'$, where F and H are species of structures.

Exercise 2.37. (see [183]) Verify that for any $m \geq 1$, the combinatorial differential equation

$$Y' = 3(m-1)\, X^2, \quad Y[\emptyset] = \emptyset,$$

possesses the m non-isomorphic solutions

$$Y = 3k\, \mathcal{C}_3 + (m - 1 - k)\, X^3, \quad k = 0, 1, \ldots, m - 1,$$

where $\mathcal{C}_3$ denotes the species of oriented cycles on 3 elements sets.

Exercises for Section 2.3

Exercise 2.38. Show that the pointing operation satisfies the following rules:

a) $(F + G)^\bullet = F^\bullet + G^\bullet$ (additivity),

b) $F \cdot G)^\bullet = F^\bullet \cdot G + F \cdot G^\bullet$ (product rule),

c) $(F \circ G)^\bullet = (F' \circ G) \cdot G^\bullet$ (chain rule for pointing).

Exercise 2.39. a) Show, by cutting off the head of non-degenerate vertebrates (see Example 3), that $\mathcal{V} = \mathcal{A} + \mathcal{V} \cdot \mathcal{A}$.

b) Deduce, for $n \geq 1$, the identity

$$n^n = \sum_{k=0}^{n-1} \binom{n}{k} k^k (n-k)^{n-k-1}.$$

Exercise 2.40. For the species Bal, of ballots (see Example 2.12), and Oct, of octopuses (see Exercises 1.7 and 2.23), establish the isomorphisms

a) $\text{Bal}^\bullet = \text{Bal}^2 \cdot E^\bullet$,

b) $\text{Oct}^\bullet + L(2X) = L(X) \cdot L(2X)$.

Exercise 2.41. The pointing operation of order n is defined by

$$F^{\bullet n} = (XD)^n F, \text{ where } D = d/dX.$$

a) Show that $F^{\bullet n} = \sum_{k=0}^{n} S(n,k) X^k F^{(k)}$, where the $S(n,k)$ are the Stirling numbers of the second kind (see Exercise 2.15).

   Hint: Use mathematical induction to show that

$$(XD)^n = \sum_{k=0}^{n} S(n,k) X^k D^k.$$

b) Express $X^n D^n$ with the help of the $(XD)^k$, $0 \leq k \leq n$ and the Stirling numbers of the first kind $s(n,k)$.

Exercise 2.42. Establish formulas (2.23), (2.24), and (2.25) giving the expected number of connected components of a random F-structure on n vertices in the cases $F = \text{Par}, \mathcal{S}$ and $\mathcal{G}$.

Exercise 2.43.   a) Let $F = G(H)$ and let n be an integer ≥ 0. Consider the random variable $\theta_n =$ the number of members of a G-assembly of random H-structures (i.e., of an F-structure) on $[n]$. Show that the expectation and the variance of θ_n are respectively given by

   i) $\text{E}(\theta_n) = \dfrac{|G^\bullet(H)[n]|}{|F[n]|}$,

   ii) $\text{Var}(\theta_n) = \dfrac{|G^{\bullet\bullet}(H)[n]|}{|F[n]|} - \left(\dfrac{|G^\bullet(H)[n]|}{|F[n]|}\right)^2$.

b) What happens to the preceding formulas in the case of cyclic assemblies (i.e., $G = \mathcal{C}$, the species of oriented cycles)?

Exercise 2.44.   a) Show that the Cartesian product of species of structures possesses the following properties: for all species F, G and H,

   i) $(F \times G) \times H = F \times (G \times H)$, (associativity)

     ii) $F \times G = G \times F$, (commutativity)

     iii) $E \times F = F \times E = F$, (neutral element)

     iv) $F \times (G + H) = F \times G + F \times H$, (distributivity)

     v) $(F \times G)^\bullet = F^\bullet \times G = F \times G^\bullet$,

b) Compare the following species

     i) $(F \times G)'$, $F' \times G$, and $F \times G'$,

     ii) $F \times (G \cdot H)$ and $(F \times G) \cdot (F \times H)$,

     iii) $(F \times G) \cdot H$ and $(F \cdot H) \times (G \cdot H)$,

     iv) $(F \times G) \circ H$ and $(F \circ H) \times (G \circ H)$.

c) Show that the passage to the generating series and the cycle index series is compatible with the Cartesian product (see Proposition 2.38).

Exercise 2.45. Describe an isomorphism between the species $L \times L$ and $S \times L$. Deduce that the law of simplification is not valid for the Cartesian product.

Exercise 2.46. Show the combinatorial equalities

a) $C_3 \times C_3 = 2\,C_3$,

b) $X^3 \times C_3 = 2\,X^3$,

c) $(X \cdot E_2) \times (X \cdot E_2) = X \cdot E_2 + X^3$,

d) $\mathcal{C} \times \wp = \mathrm{Oct} + \mathcal{C}$.

Exercise 2.47.   a) For each $k \geq 0$, consider the species $\wp^{[k]}$ of subsets of cardinality k:

$$\wp^{[k]}[U] = \{V \mid V \subseteq U, |V| = k\}.$$

   Establish the combinatorial equality

$$\wp^{[m]} \times \wp^{[n]} = \sum_{k=0}^{\min(m,n)} E_{m-k} \cdot E_k \cdot E_{n-k} \cdot E$$

$$= \sum_{k=0}^{\min(m,n)} E_{m-k} \cdot E_{n-k} \cdot \wp^{[k]}.$$

b) A lottery proceeds in the following manner: the "player" (resp., the "house") chooses a set V (resp., W) of k integers among the integers 1 to N. The player gains the pot i if $|V \cap W| = i$. In the case where $N = 49, k = 6$, calculate the probability that the player wins the pot i, for $i = 2, 3, 4, 5, 6$.

Exercises for Section 2.4

Exercise 2.48. Establish directly formula (2.31) giving the number $(\wp^{[2]}[\sigma])_k$ for a given permutation σ of cycle type $(\sigma_1, \sigma_2, \sigma_3, \dots)$, by considering separately the cycles of length k of $\wp^{[2]}[\sigma]$ originating from either two different cycles of σ, or from the same cycle of σ.

Exercise 2.49. (See [74].) Show that for any permutation σ of cycle type $(\sigma_1, \sigma_2, \dots)$ and for all $k \geq 1$ and $m \geq 1$,

a) $\displaystyle \sum_{d|k} \mu(k/d)(\sigma^d)_1^m = \sum_{[j_1,\dots,j_m]=k} j_1 \dots j_m \; \sigma_{j_1} \dots \sigma_{j_m},$

b) $\displaystyle (\sigma^k)_m = \sum_{d|k,(m,k/d)=1} d \; \sigma_{dm},$

c) $\displaystyle \frac{1}{k} \sum_{d|k} \mu(k/d)(\sigma^d)_m = \sum_{d|(m,k)} \frac{\mu(d)}{d} \; \sigma_{mk/d}.$

Exercise 2.50. Show that

a) for any permutation β and any $\omega \geq 1$, with $\beta^\omega = Id$,

$$\sum_{k \geq 1} \beta_k = \frac{1}{\omega} \sum_{d|\omega} \phi(\omega/d) \text{fix} \, \beta^d,$$

b) for any species of structures G and any permutation σ of order ω,

$$\text{fix} \, (\wp \square G)[\sigma] = 2^{(1/\omega) \sum_{d|\omega} \phi(\omega/d) \, \text{fix} \, G[\sigma^d]},$$

where ϕ denotes Euler's ϕ-function.

Exercise 2.51. Denote by $\mathcal{G}_l$, the species of (undirected) graphs with at most one loop at each vertex and by $\mathcal{D}_\mathrm{w}$, the species of directed graphs without loops.

a) Verify the following relations:

$$\text{i) } \mathcal{G}_l = \wp \times \mathcal{G}, \qquad \text{ii) } \mathcal{D} = \wp \times \mathcal{D}_\mathrm{w}.$$

b) Determine $\text{fix} \, \mathcal{G}_l[\sigma]$ and $\text{fix} \, \mathcal{D}_\mathrm{w}[\sigma]$.

Exercise 2.52. Write a program permitting the calculation of the first terms of the cycle index series and the type generating series of the following species:

$$\text{a) } \mathcal{G}, \qquad \text{b) } \mathcal{G}_l, \qquad \text{c) } \mathcal{D}, \qquad \text{d) } \mathcal{D}_\mathrm{w}.$$

Exercise 2.53. Starting from the definitions, show that

a) the operation $\square$ is associative,

b) the species $E^{\bullet}$ of pointed sets is the neutral element for the operation $\square$,

c) the operation $\square$ is distributive on the right by the operation of Cartesian product $\times$: $(F \times G) \square H = (F \square H) \times (G \square H)$,

d) the operation $\square$ is not distributive on the left by $\times$ (give an example illustrating this fact),

e) the Cartesian product squared $F \times F$ can be expressed with the help of the ordinary product and functorial composition: $F \times F = ((X + X^2) \cdot E) \square F$.

Exercise 2.54. Calculate the following series $(\mathcal{C} \square \mathcal{C}_p)(x)$, $(\widetilde{\mathcal{C} \square \mathcal{C}_p})(x)$ and $Z_{\mathcal{C} \square \mathcal{C}_p}$, where $\mathcal{C}_p$ denotes the species of oriented cycles restricted to the cardinality p, a prime number.

Exercise 2.55. Let F and G be any species of structures.

a) Compare the species $(F \cdot E) \square G$ and $F \square (E \cdot G)$.

b) If $G(0) \neq 0$, show that $F \circ G$ is a subspecies of $(F \cdot E) \square (G \cdot E)$.

c) What happens in b) in the case of the species of graphs (see formula (2.27))?

Exercise 2.56. Let U be a finite set. A **covering** of U is a set of non-empty subsets of U whose union is U. Consider the species Cov (resp., Cov$^{[m]}$) of coverings of sets (resp., coverings of sets by exactly m non-empty subsets).

a) Show that $E \cdot \text{Cov} = \wp \square \wp^{+}$, and that $E \cdot \text{Cov}^{[m]} = \wp^{[m]} \square \wp^{+}$, where $\wp^{+}$ is the species of non-empty subsets.

b) If $\mid U \mid = n$, show that

i) $\mid \text{Cov}[U] \mid = \displaystyle\sum_{i=0}^{n} (-1)^i \binom{n}{i} 2^{2^{n-i}-1}$,

ii) $\mid \text{Cov}^{[m]}[U] \mid = \displaystyle\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{2^{n-i}-1}{m}$.

Exercise 2.57. By definition, a **simplicial complex** on a finite set U is a collection C of subsets of U, called **simplices**, such that $u \in U$ implies that $\{u\} \in C$, and $\emptyset \neq T \subseteq S \in C$ implies $T \in C$. The **dimension** of a complex C is the maximum dimension of its simplices, the dimension of a simplex $S \in C$ being $|S| - 1$. Let $m \geq 0$; a complex C is called a pure m-complex if all of its maximal simplices are of dimension m. Show that the species $C^{[m]}$ of pure m-complexes satisfies

a) $C^{[m]} = \wp \square \wp^{[m+1]}$,

b) fix $C^{[m]}[\sigma] = 2^{P_m(\sigma_1, \sigma_2, \dots)}$, where $P_m \in \mathbb{Z}[\sigma_1, \sigma_2, \dots]$ is a polynomial of total degree $m + 1$ in the variables $\sigma_1, \sigma_2, \dots$.

Exercise 2.58. (see [74]) Let F be any species and m, an integer ≥ 0. Define the species $F^{<m>}$ of F-**structured words of length** m by setting, for any finite set V,

$$F^{<m>}[V] = (F[m] \times \Phi[m, V])/\mathcal{S}_m,$$

where $\Phi[m, V] = \{\varphi \mid \varphi : [m] \to V\}, \mathcal{S}_m$ denotes the symmetric group of order m and the quotient set above is interpreted as the set of orbits of the action

$$\mathcal{S}_m \times (F[m] \times \Phi[m, V]) \to (F[m] \times \Phi[m, V]),$$

defined by $\beta \cdot (s, \varphi) = (F[\beta](s), \varphi \circ \beta^{-1})$. The elements of $F^{<m>}[V]$ are called the F-structured words of length m on the alphabet V or also the types of V-colored F-structures of order m. In particular, taking $F = L, E, \mathcal{C}$, $L^{<m>}$ is the species of words of length m, $E^{<m>}$ is the species of abelian words of length m, and $\mathcal{C}^{<m>}$ is the species of circular words of length m.

a) Prove that for any permutation $\sigma : V \to V$,

$$\text{fix } F^{<m>}[\sigma] = Z_{F_m}(\text{fix } \sigma, \text{fix } \sigma^2, \text{fix } \sigma^3 \dots).$$

b) Deduce expressions for the series $F^{<m>}(x)$ and $\widetilde{F^{<m>}}(x)$.

Exercise 2.59. Every m-ary relation on U can be considered as a **set of words of length** m **on the alphabet** U, whence the combinatorial equation $\text{Rel}^{[m]} = \wp \square L^{<m>}$, where $L^{<m>}$ is the species of words of length m. More generally, given a species F, define the species $\text{Rel}_F^{[m]}$ of m-ary F-structured relations by $\text{Rel}_F^{[m]} = \wp \square F^{<m>}$, where $F^{<m>}$ is defined in the preceding exercise. Prove that for any permutation $\sigma : U \to U$:

a) fix $\text{Rel}^{[m]}[\sigma] = 2^{\sum_{i_1, \dots, i_m \geq 1} i_1 \dots i_m \sigma_{i_1} \dots \sigma_{i_m}/[i_1, \dots, i_m]}$,

b) fix $\text{Rel}_F^{[m]}[\sigma] = 2^{\frac{1}{\omega} \sum_{d \mid \omega} \phi(\omega/d) Z_{F_m}(f_d, f_{2d}, \dots)}$, where $f_k = \text{fix } \sigma^k = \sum_{d \mid k} d\sigma_d$, ω is the order of σ, and ϕ denotes Euler's ϕ-function.

Chapter 3

Variations on the theme of Species

3.1 Weighted species

In enumerative combinatorics, it is often required to consider some parameters related to the characteristics of the structures. For instance, in computer science, the complexity analysis of algorithms often involves the enumeration of structures according to certain descriptive parameters such as the number of leaves or the depth of binary rooted trees. It is this kind of enumeration problem that we are now going to address through the introduction of a variant of the concept of species of structures: **weighted species**. All constructions introduced in the preceding chapters and sections will be extended by taking into account this addition of weighting.

Example 3.1. Assign to each rooted tree $\alpha \in \mathcal{A}[U]$ a weight $w(\alpha)$ by setting

$$w(\alpha) = t^{f(\alpha)}, \tag{3.1}$$

where t is a formal variable and $f(\alpha)$ denotes the number of leaves of α (see Figure 3.1). This permits the regrouping of rooted trees according to the descriptive parameter "number of leaves". We say that the set $\mathcal{A}[U]$ of all rooted trees on U is "weighted" by (3.1) and also that the variable t acts as a leaf "counter". The "inventory" of rooted trees on U according to this weight w, denoted by $|\mathcal{A}[U]|_w$, is defined as the sum of the weights $w(\alpha)$, $\alpha \in \mathcal{A}[U]$:

$$|\mathcal{A}[U]|_w = \sum_{\alpha \in \mathcal{A}[U]} w(\alpha) = \sum_{\alpha \in \mathcal{A}[U]} t^{f(\alpha)}.$$

Regrouping terms according to the powers of t gives a polynomial in t. If $|U| = n$, we have

$$|\mathcal{A}[U]|_w = a_n(t) = \sum_{k=0}^{n} a_{n,k} t^k.$$

The coefficient $a_{n,k}$ gives the number of rooted trees on n elements of which k are leaves. This inventory is more refined than the simple enumeration of rooted trees; the substitution $t := 1$ has the effect of giving weight 1 to each rooted tree and we obtain $a_n(1) = |\mathcal{A}[U]| = n^{n-1}$, for $n \geq 1$.
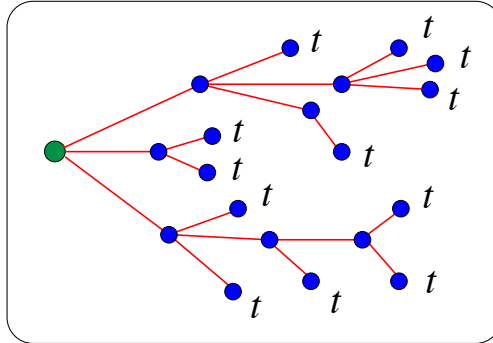
Figure 3.1: A rooted tree α with $w(\alpha) = t^{12}$.

One can equally be interested in the enumeration of unlabeled rooted trees, and also of rooted trees left fixed by a given permutation, according to the same parameter "number of leaves". It is for this purpose that the concept of species of weighted structures is introduced.

Note that the weight (3.1) is a function $w : \mathcal{A}[U] \longrightarrow \mathcal{R}$, where $\mathcal{R}$ is a polynomial ring in the variable t. The pair $(\mathcal{A}[U], w)$ is said to be a weighted set in the ring $\mathcal{R}$ ($\mathcal{R}$-weighted, for short). Let us first study the general concept of $\mathcal{R}$-weighted sets as well as related constructions.

Let $\mathbb{K} \subseteq \mathbb{C}$ be an integral domain (for example $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$) and $\mathcal{R}$, a ring of formal power series in an arbitrary number of variables, with coefficients in $\mathbb{K}$.

Definition 3.2. An **$\mathcal{R}$-weighted set** is a pair (A, w), where A is a (finite or infinite) set and $w : A \longrightarrow \mathcal{R}$ is a function which associates a **weight** $w(\alpha) \in \mathcal{R}$ to each element $\alpha \in A$. If the following sum, denoted by $|A|_w$, exists, the weighted set (A, w) is said to be **summable** (for a precise definition, see Exercise 3.1) and $|A|_w$ is called the **inventory** (or **total weight** or **cardinality**) of the weighted set (A, w): $|A|_w = \sum_{\alpha \in A} w(\alpha)$.

Many set-theoretic constructions can be extended to $\mathcal{R}$-weighted sets.

Definition 3.3. Let (A, w) and (B, v) be $\mathcal{R}$-weighted sets. A **morphism of $\mathcal{R}$-weighted sets** $f : (A, w) \longrightarrow (B, v)$, is a function $f : A \longrightarrow B$ compatible with the weighting (one also says that the function f is **weight preserving**), that is to say, such that $w = v \circ f$. Moreover, if f is a bijection, f is called an **isomorphism of weighted sets** and we write $(A, w) \simeq (B, v)$. Observe that

$$(A, w) \simeq (B, v) \quad \Longrightarrow \quad |A|_w = |B|_v \, .$$

Definition 3.4. Let (A, w) and (B, v) be $\mathcal{R}$-weighted sets. Define

i) The **sum** $(A, w) + (B, v)$ as the $\mathcal{R}$-weighted set $(A + B, \mu)$, where $A + B$ denotes the disjoint union of the sets A and B and μ is the weight function defined by

$$\mu(x) = \begin{cases} w(x), & \text{if } x \in A, \\ v(x), & \text{if } x \in B. \end{cases}$$

ii) The **product** $(A, w) \times (B, v)$ as the $\mathcal{R}$-weighted set $(A \times B, \rho)$, where $A \times B$ denotes the cartesian product of sets A and B and ρ is the weight function defined by $\rho(x, y) = w(x)v(y)$.

The proof of the following proposition is left as an exercise.

Proposition 3.5. *Let (A, w) and (B, v) be $\mathcal{R}$-weighted sets. With the preceding notation and conventions, we have*

$$\begin{aligned} &\text{a) } |A + B|_\mu = |A|_w + |B|_v, \\ &\text{b) } |A \times B|_\rho = |A|_w |B|_v. \end{aligned} \tag{3.2}$$

To every finite set A, one can associate an $\mathcal{R}$-weighted set (A, w) by giving each element $\alpha \in A$ the weight $w(\alpha) = 1 \in \mathcal{R}$. This weighting is called **trivial**. Then $|A|_w = |A|$. With this notion at hand, we can now introduce the notion of weighted species. It constitutes an important variant of the concept of species of structures, which allows a more refined enumeration of structures and their classification according to various descriptive parameters, by the addition of well-chosen weights.

Definition 3.6. Let $\mathcal{R}$ be a ring of formal power series or of polynomials over a ring $\mathbb{K} \subseteq \mathbb{C}$. An $\mathcal{R}$-**weighted species** is a rule F which produces

i) for each finite set U, a finite or summable $\mathcal{R}$-weighted set $(F[U], w_U)$,

ii) for each bijection $\sigma : U \longrightarrow V$, a function $F[\sigma] : (F[U], w_U) \longrightarrow (F[V], w_V)$ preserving the weights (i.e., a weighted set morphism).

Moreover, the functions $F[\sigma]$ must satisfy the following functoriality properties:

a) if $\sigma : U \longrightarrow V$ and $\tau : V \longrightarrow W$ are bijections, then $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$,

b) for each set U, if Id_U denotes the identity bijection of U to U, then $F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$.

As before, an element $s \in F[U]$ is called an F-structure on U, and the function $F[\sigma]$, the transport of F-structures along σ.

It follows from the definition that the transport of structures $F[\sigma]$ along a bijection $\sigma : U \longrightarrow V$ is a weight preserving bijection and that $|F[U]|_{w_U} = |F[V]|_{w_V}$. Two F-structures s_1 and s_2 are called isomorphic (i.e., have same type) if they are transportable one on the other along a bijection σ. As $F[\sigma]$ preserves weights, s_1 and s_2 are then forced to have the same weight. This permits the weighting of the set $F[U]/\sim$, of isomorphism types of F-structures (i.e., unlabeled F-structures), by defining the weight of a type as the weight of an arbitrary structure representing this type. To be more precise, it is useful to write $F = F_w$ to denote the weighted species F together with the family of all the weight functions $w_U : F[U] \longrightarrow \mathcal{R}$ associated to F.

Example 3.7.) Consider the species $\mathcal{S}_w$ of permutations with cycle counter α, i.e., the weight $w(\sigma)$ of a permutation σ is $w(\sigma) = \alpha^{\text{cyc}(\sigma)}$, where α is a formal variable and $\text{cyc}(\sigma)$ is the number of cycles of σ. This weighting has values in the ring of polynomials $\mathbb{Z}[\alpha]$, so $\mathcal{S}_w$ is a $\mathbb{Z}[\alpha]$-weighted species.

Observe that every species of structures F can be identified with an $\mathcal{R}$-weighted species by considering that the sets $F[U]$ are all provided with the trivial weighting. Moreover, this identification is compatible with all the operations and the passage to the diverse generating series introduced below.

Definition 3.8. Let $F = F_w$ be an $\mathcal{R}$-weighted species of structures. The generating series of F is the exponential formal power series $F_w(x)$ with coefficients in $\mathcal{R}$ defined by

$$F_w(x) = \sum_{n \geq 0} |F[n]|_w \frac{x^n}{n!},$$

where $|F[n]|_w$ is the inventory of the set of F-structures on $[n]$. Its cycle index series is defined by

$$Z_{F_w}(x_1, x_2, x_3, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathcal{S}_n} |\text{Fix}\, F[\sigma]|_w \, x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots \right),$$

where as before, σ_i is the number of cycles of length i in σ. Note that the set $\text{Fix}\, F[\sigma]$ inherits the weighting on $F[U]$ of which it is a subset, whence $|\text{Fix}\, F[\sigma]|_w$ is the inventory of all the F-structures on $[n]$ left fixed under the transport $F[\sigma]$. Since the weighting is preserved by transport of structures, we can define the **type generating series** of F_w:

$$\widetilde{F_w}(x) = \sum_{n \geq 0} \left| F[n]_{/\sim} \right|_w x^n,$$

where $\sim$ is the isomorphism relation. Hence, $|F[n]/\sim|_w$ is the inventory of unlabeled F-structures on n points.

As in the non-weighted case, by suitably regrouping terms, the cycle index series can be written as

$$Z_{F_w}(x_1, x_2, \ldots) = \sum_{n_1 + 2n_2 + \ldots < \infty} \left| \text{Fix}\, F[n_1, n_2, \ldots] \right|_w \frac{x_1^{n_1} x_2^{n_2} \cdots}{1^{n_1} n_1! 2^{n_2} n_2! \ldots},$$

where $|\text{Fix}\, F[n_1, n_2, \ldots]|_w$ denotes the inventory of the set F-structures left fixed under the action of a permutation σ of type $(n_1, n_2, n_3, \ldots)$. Moreover, the formulas

$$\text{a) } F_w(x) = Z_{F_w}(x, 0, 0, \ldots) \quad \text{and} \quad \text{b) } \widetilde{F_w}(x) = Z_{F_w}(x, x^2, x^3, \ldots)$$

remain valid.

Species	Structure	Weight
$F_w + G_v$	s	$w(s)$ if $s \in F[U]$ $v(s)$ if $s \in G[U]$
$F_w \cdot G_v$	$s = (f, g)$	$w(f)v(g)$
$F_w \circ G_v$	$s = (\pi, f, (\gamma_p)_{p \in \pi})$	$w(f) \prod_{p \in \pi} v(\gamma_p)$
F'_w	s	$w(s)$
$F_w{}^{\bullet}$	$s = (f, u)$	$w(f)$
$F_w \times G_v$	$s = (f, g)$	$w(f)v(g)$
$F_w \square G$ (G non-weighted)	s	$w(s)$

Table 3.1: Weights for usual operations

The operations $+$, $\cdot$, $\circ$, $'$, $^{\bullet}$, $\times$ and $\square$ are defined in the same fashion on weighted species as in the non-weighted case, but the weights of the structures have to be carefully defined. See Table 3.1 for the precise definitions. Observe that for the product, the substitution, and the Cartesian product, a principle of multiplicativity, induced from (3.2), is used to define the weights. The goal, of course, is to reflect correctly the corresponding operations on the generating series. However, the plethystic substitution of the cycle index series must undergo an important modification in the weighted case.

Definition 3.9. Let F_w and G_v be two weighted species of structures, such that $G(0) = 0$ (i.e., there is no G-structure on the empty set). The **plethystic substitution** of Z_{G_v} in Z_{F_w}, denoted by $Z_{F_w} \circ Z_{G_v}$ (or $Z_{F_w}(Z_{G_v})$) is defined by

$$Z_{F_w} \circ Z_{G_v} = Z_{F_w}((Z_{G_v})_1, (Z_{G_v})_2, (Z_{G_v})_3, \ldots), \qquad (3.3)$$

where, for $k = 1, 2, 3, \ldots$, $(Z_{G_v})_k (x_1, x_2, x_3, \ldots) = Z_{G_{v^k}}(x_k, x_{2k}, x_{3k}, \ldots)$.

The reader should take note that in the series $Z_{G_{v^k}}(x_k, x_{2k}, x_{3k}, \ldots)$, the weighting v is raised to the power k (in the ring $\mathcal{R}$) and the indices of the variables x_i are multiplied by k.

Remark 3.10. It often happens that the weights of the structures of a species G_v are monomials in the variables α, β, γ, ... In this case, setting $g(\alpha, \beta, \gamma, \ldots; x_1, x_2, x_3, \ldots) = Z_{G_v}(x_1, x_2, x_3, \ldots)$,

$$g_k = (Z_{G_v})_k = g(\alpha^k, \beta^k, \gamma^k, \ldots; x_k, x_{2k}, x_{3k}, \ldots).$$

Proposition 3.11. *Let F_w and G_v be two $\mathcal{R}$-weighted species of structures. Then*

$$
\begin{aligned}
&\text{a) } (F_w + G_v)(x) = F_w(x) + G_v(x), &&\text{a') } Z_{(F_w+G_v)} = Z_{F_w} + Z_{G_v}; \\
&\text{b) } (F_w G_v)(x) = F_w(x) G_v(x), &&\text{b') } Z_{(F_w \cdot G_v)} = Z_{F_w} Z_{G_v}; \\
&\text{c) } F'_w(x) = \frac{d}{dx} F_w(x), &&\text{c') } Z_{F'_w} = \frac{\partial}{\partial x_1} Z_{F_w}; \\
&\text{d) } (F_w \times G_v)(x) = F_w(x) \times G_v(x), &&\text{d') } Z_{(F_w \times G_v)} = Z_{F_w} \times Z_{G_v}; \\
&\text{e) } (F_w \circ G_v)(x) = F_w(G_v(x)), &&\text{e') } Z_{F_w \circ G_v} = Z_{F_w} \circ Z_{G_v}; \\
&\text{f) } (F_w \,\square\, G)(x) = F_w(x) \,\square\, G(x), &&\text{f') } Z_{(F_w \,\square\, G)} = Z_{F_w} \,\square\, Z_G.
\end{aligned}
\tag{3.4}
$$

It is necessary to assume, for e) and e'), that $G_v(0) = 0$; and, for f), that G is an ordinary (non-weighted) species.

These equalities constitute powerful tools for calculation. Their proof is left as an exercise to the reader, except for formula (3.4), e') which is proved in Chapter 4 of [22].

Let α be a fixed element of the ring $\mathcal{R}$. To each species F, we can associate the $\mathcal{R}$-weighted species F_α by giving each structure of $F[U]$ the same weight α. Clearly then $F_\alpha(x) = \alpha F(x)$, and $Z_{F_\alpha} = \alpha Z_F$.

Example 3.12. In this manner, starting from $\mathcal{C}$, the species of oriented cycles, we can construct the species C_α with weight α for each cycle. The species $\mathcal{S}_w$, of Example 3.7, is then isomorphic to the species $E(\mathcal{C}_\alpha)$, by the principle of multiplicativity. We deduce then that

$$
\begin{aligned}
&\text{a) } \mathcal{S}_w(x) = \exp(-\alpha \log(1-x)) = \left(\frac{1}{1-x}\right)^\alpha, \\
&\text{b) } \widetilde{\mathcal{S}_w}(x) = \prod_{k\geq 1} \frac{1}{1 - \alpha x^k} = \prod_{k\geq 1} \left(\frac{1}{1-x^k}\right)^{\nu_k(\alpha)}, \\
&\text{c) } Z_{\mathcal{S}_w}(x_1, x_2, x_3, \ldots) = \prod_{k\geq 1} \left(\frac{1}{1-x_k}\right)^{\nu_k(\alpha)},
\end{aligned}
\tag{3.5}
$$

where $\nu_n(\alpha) = \frac{1}{n} \sum_{d|n} \phi(d) \alpha^{n/d}$ and ϕ denotes Euler's ϕ-function.

Example 3.13. In a similar fashion, considering the species E_{+t} of non-empty sets of weight t, we can form the species $\text{Par}_w = E(E_{+t})$ of partitions weighted by number of parts: $w(\pi) = t^{|\pi|}$, where $|\pi|$ is the number of blocks of a partition π. The following series are then obtained

$$
\begin{aligned}
&\text{a) } \text{Par}_w(x) = \exp t(e^x - 1), \\
&\text{b) } \widetilde{\text{Par}_w}(x) = \prod_{k\geq 1} \frac{1}{1 - tx^k} = \prod_{k\geq 1} \left(\frac{1}{1-x^k}\right)^{\nu_k(t)}, \\
&\text{c) } Z_{\text{Par}_w}(x_1, x_2, x_3, \ldots) = \exp \sum_{k\geq 1} \frac{1}{k} t^k \left(\exp(x_k + \frac{1}{2}x_{2k} + \frac{1}{3}x_{3k} + \ldots) - 1\right).
\end{aligned}
\tag{3.6}
$$

Example 3.14. Consider now the species $\mathcal{A}_s$ of rooted trees each having weight s. By substitution in $\mathcal{S}_w$, we obtain the species $\mathrm{End}_v = \mathcal{S}_w \circ \mathcal{A}_s$ of endofunctions ψ, weighted by $v(\psi) = s^{\mathrm{rec}(\psi)}\alpha^{\mathrm{cyc}(\psi)}$, where $\mathrm{rec}(\psi)$ denotes the number of recurrent elements of ψ, and $\mathrm{cyc}(\psi)$ denotes the number of connected components of ψ. Since $\mathcal{A}_s(x) = s\mathcal{A}(x)$, $\widetilde{\mathcal{A}_s}(x) = s\widetilde{\mathcal{A}}(x)$, and $Z_{\mathcal{A}_s} = sZ_{\mathcal{A}}$, we obtain, after some calculation, the following series

$$
\begin{aligned}
&\text{a) } \mathrm{End}_v(x) = \left(\frac{1}{1 - s\mathcal{A}(x)}\right)^{\alpha}, \\
&\text{b) } \widetilde{\mathrm{End}_v}(x) = \prod_{k\geq 1}\left(\frac{1}{1 - s^k\widetilde{\mathcal{A}}(x^k)}\right)^{\nu_k(\alpha)}, \\
&\text{c) } Z_{\mathrm{End}_v}(x_1, x_2, x_3, \ldots) = \prod_{k\geq 1}\left(\frac{1}{1 - s^k Z_{\mathcal{A}}(x_k, x_{2k}, x_{3k}, \ldots)}\right)^{\nu_k(\alpha)}.
\end{aligned}
\tag{3.7}
$$

Expanding the series $\mathrm{End}_v(x)$ according to powers of x, the following expression is obtained for the total weight of all endofunctions on a set of $n \geq 1$ elements (see Exercise 3.5):

$$
|\mathrm{End}\,[n]|_v = \sum_{k=1}^{n}\frac{k}{n}\binom{n}{k}n^{n-k}s^k\alpha(\alpha+1)\ldots(\alpha+k-1).
\tag{3.8}
$$

In particular, setting $s = \alpha = 1$, we deduce the identity

$$
n^n = \sum_{k=1}^{n} kn^{n-k}(n-1)(n-2)\ldots(n-k+1).
$$

Example 3.15. SYMMETRIC FUNCTIONS. Substituting the species X_t of singletons of weight t into an ordinary species F, we obtain the weighted species $F(X_t)$. Here, the weight of a structure of species $F(X_t)$ on a set U is given by $t^{|U|}$. Since an $F(X_t)$-structure is simply an F-structure on a set U together with a weight t on each of its elements, we have the identity $F(X_t) = F \times E(X_t)$. More generally, let τ be a finite sequence of distinct variables $t_1, t_2, \ldots, t_k$, and define the weighted species $X_\tau = X_{t_1} + X_{t_2} + \ldots + X_{t_k}$. Thus an X_τ-structure is a singleton of weight t_i, for some i between 1 and k. Now consider the composite species $F(X_\tau) = F(X_{t_1} + X_{t_2} + \ldots + X_{t_k})$; an $F(X_\tau)$-structure corresponds to placing an F-structure on a set of singletons of weights t_i, $1 \leq i \leq k$. Figure 3.2 shows that

$$
F(X_{t_1} + X_{t_2} + \ldots + X_{t_k}) = F \times (E(X_{t_1})E(X_{t_2})\ldots E(X_{t_k})).
$$

Another way of visualizing $F(X_\tau)$-structures consists of assigning a "color" i, $1 \leq i \leq k$, to each element of the underlying set of an F-structure and to give to this F-structure the weight $t_1^{n_1}t_2^{n_2}\ldots t_k^{n_k}$, where n_i is the number of elements of color i in the underlying set. The general formula of weighted plethysm (3.4), e'), then gives

$$
Z_{F(X_{t_1}+X_{t_2}+\ldots+X_{t_k})} = Z_F\big(x_1(t_1 + t_2 + \ldots), x_2(t_1^2 + t_2^2 \ldots), \ldots\big)
$$

since

$$
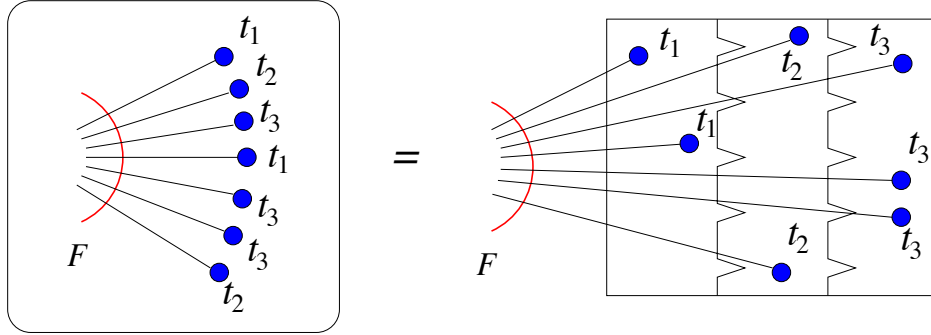\Big(Z_{X_{t_1}+X_{t_2}+\ldots+X_{t_k}}\Big)_i = x_i(t_1^i + t_2^i + \ldots).
$$

Figure 3.2: Decomposition respective to type of weight.

This cycle index series allows, in particular, the enumeration of unlabeled $F(X_\tau)$-structures. By restriction to n-element sets (i.e., by considering the species F_n), the enumeration of unlabeled F-structures on n points, colored with k colors, is given by the following polynomial in k variables $t_1, t_2, \ldots, t_k$

$$[x^n]\, \widetilde{F(X_\tau)}(x) = Z_{F_n}\big((t_1 + \ldots + t_k), (t_1^2 + \ldots + t_k^2), \ldots, (t_1^n + \ldots + t_k^n)\big), \qquad (3.9)$$

of total degree n. As is studied in Chapter 4 of [22], this is essentially the classic enumeration theorem of Pólya. Moreover, the functions $p_n = t_1^n + t_2^n + t_3^n + \ldots$, are the traditional **power sum** symmetric functions, so that for each n and each species F, we obtain a symmetric function $Z_{F_n}(p_1, p_2, p_3, \ldots)$ by enumeration of unlabeled $F(X_\tau)$-structures on a set of cardinality n. In particular, for the species E_n of sets of cardinality n, we obtain the complete homogeneous symmetric functions

$$
\begin{aligned}
h_n(t_1, t_2, t_3, \ldots) &= [x^n]\, \widetilde{E(X_\tau)}(x) \\
&= \sum_{n_1 + n_2 + \ldots n_k = n} t_1^{n_1}\, t_2^{n_2} \cdots t_k^{n_k},
\end{aligned}
$$

and the following well known expression for complete homogeneous symmetric functions in terms of power sums

$$h_n = Z_{E_n}(p_1, p_2, p_3 \ldots) = \sum_{\mathbf{d} \vdash n} \frac{p_1^{d_1}\, p_2^{d_2}\, p_3^{d_3} \cdots}{\mathrm{aut}(\mathbf{d})},$$

the sum being taken over all the sequences $\mathbf{d} = (d_1, d_2, \ldots)$ of integers ≥ 0 such that $n = d_1 + 2d_2 + 3d_3 + \ldots$. Equivalently

$$\sum_{n \geq 0} h_n = \widetilde{E(X_\tau)}(x)\big|_{x=1} = Z_E(p_1, p_2, \ldots) = \exp\left(\sum_{k \geq 1} \frac{p_k}{k}\right).$$

Note that it is possible to take an infinite number of variables $t_1, t_2, \ldots$ in these formulas. In this case, the species $F(X_\tau)$ is no longer finite, but it is summable, that is to say, for any finite set

U, $F(X_\tau)[U]$ is a summable weighted set in $\mathcal{R} = \mathbb{C}[\![t_1, t_2, \ldots]\!]$. Since the power sum functions (in infinitely many variables) are algebraically independent in the ring of symmetric functions, the identities in (3.9) characterize the index series Z_F. We point out that according to Frobenius correspondence, $Z_{F_n}(p_1, p_2, p_3, \ldots)$ is the symmetric function corresponding to the representation of the symmetric group obtained by linearizing the group action $\mathcal{S}_n \times F[n] \longrightarrow F[n]$. In fact, for any permutation $\sigma \in \mathcal{S}_n$, fix $F[\sigma]$ is equal to the trace of the permutation matrix of $F[\sigma]$, that is, the character of the representation (see [297] for instance for more details). With this point of view, we can easily deduce certain symmetric function formulas from previous results on cycle index series, for example (see Exercise 3.7) we get that

$$Z_{\mathcal{S}}(p_1, p_2, \ldots) = \prod_{k \geq 1} \frac{1}{1 - p_k}$$

corresponds to the action of the symmetric group on itself by conjugation.

To conclude the present section, we show how weighted species can lead to combinatorial models for certain classical families of orthogonal polynomials, such as, for example, Hermite polynomials. See also Exercise 3.8 as well as Exercise 3.9 where Laguerre polynomials are treated.

Example 3.16. HERMITE POLYNOMIALS. Consider the weighted species Inv_w of all involutions (i.e., permutations φ such that $\varphi \circ \varphi = \text{Id}$), weighted by $w(\varphi) = t^{\varphi_1}(-1)^{\varphi_2}$, where φ_1 is the number of fixed points of φ and φ_2 is the number of cycles of length 2 (the edges in Figure 3.3) in φ. We obtain the combinatorial equation $\text{Inv}_w = E(X_t + (\mathcal{C}_2)_{-1})$, where $(\mathcal{C}_2)_{-1}$ denotes the species of cycles of length 2 and weight -1, and it immediately follows that

$$\text{Inv}_w(x) = \exp(tx - \frac{1}{2}x^2) = \sum_{n \geq 0} H_n(t) \frac{x^n}{n!}, \tag{3.10}$$

where the coefficient $H_n(t)$ denotes, by definition, the (unitary) Hermite polynomial of degree n in t. Thus this classical polynomial appears as the inventory of all involutions on a set of n elements:

$$H_n(t) = \sum_{\varphi \in Inv[n]} t^{\varphi_1}(-1)^{\varphi_2}. \tag{3.11}$$

The other series associated with Inv_w are given by

$$\widetilde{\text{Inv}_w}(x) = \frac{1}{(1 - tx)(1 + x^2)}, \quad \text{and} \quad Z_{\text{Inv}_w} = \exp \sum_{k \geq 1} \frac{1}{k} \left(t^k x_k + \frac{(-1)^k}{2}(x_k^2 + x_{2k}) \right). \tag{3.12}$$

since $Z_X = x_1$ and $Z_{\mathcal{C}_2} = Z_{E_2} = (x_1^2 + x_2)/2$. Most classical properties of Hermite polynomials (recurrences, differential equations, Mehler's formula, coefficients of linearization) can be deduced from this combinatorial model (see Exercise 3.8).
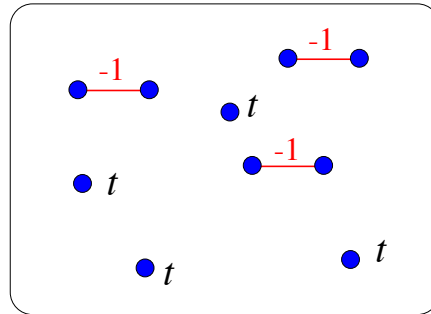
Figure 3.3: An involution of weight $-t^4$.

3.2   Extension to the multisort context

3.2.1   Multisort species

The theory of species can naturally be extended in yet another direction by considering structures constructed on sets containing several **sorts** of elements. It is an undertaking analogous to the introduction of functions in many variables.

Example 3.17. Consider tri-chromatic simple graphs, that is to say, simple graphs constructed on triplets of sets (U_1, U_2, U_3), corresponding to three distinct colors, in such a manner that adjacent vertices have different colors. Figure 3.4 represents one such graph, where $U_1 = \{a, b, c, d, e\}$, $U_2 = \{1, 2, 3, 4, 5, 6, 7\}$ and $U_3 = \{m, n, p, q\}$ correspond respectively to the colors yellow, red and blue. Another example is given by rooted trees constructed on a set having two sorts of elements:
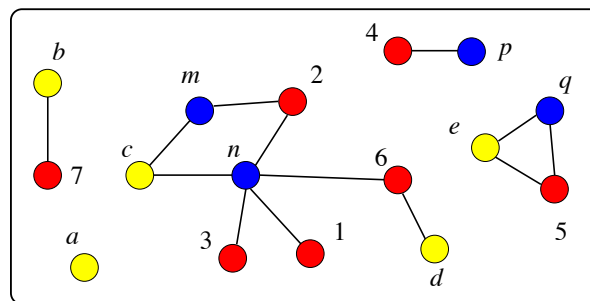

Figure 3.4: Sorts as colors.

(green) leaves and (blue) internal vertices. See Figure 3.5. In this example, the elements of leaf sort are not placed in an arbitrary manner; by convention, they are located at the end of the paths starting from the root. In other words, these are the vertices with empty "fibers".

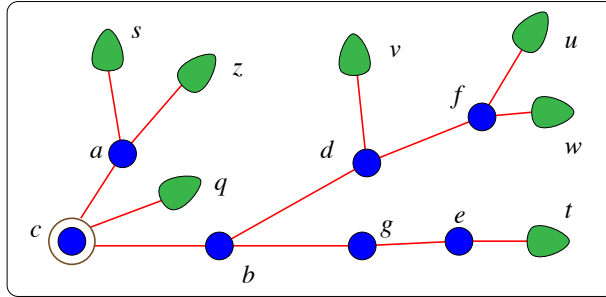For transport of structures, multisort species are distinguished by the fact that "transport is

Figure 3.5: Internal nodes and leaves as sorts.

carried out along bijections preserving the sort of the elements". We illustrate this with rooted trees. Figure 3.6 represents the transport of a rooted tree on two sorts (internal vertices and leaves). The bijection $\sigma : U_1 + U_2 \longrightarrow V_1 + V_2$ along which the transport is carried "must necessarily" send each internal vertex $(\in U_1)$ to an internal vertex $(\in V_1)$ and each leaf $(\in U_2)$ to a leaf $(\in V_2)$.
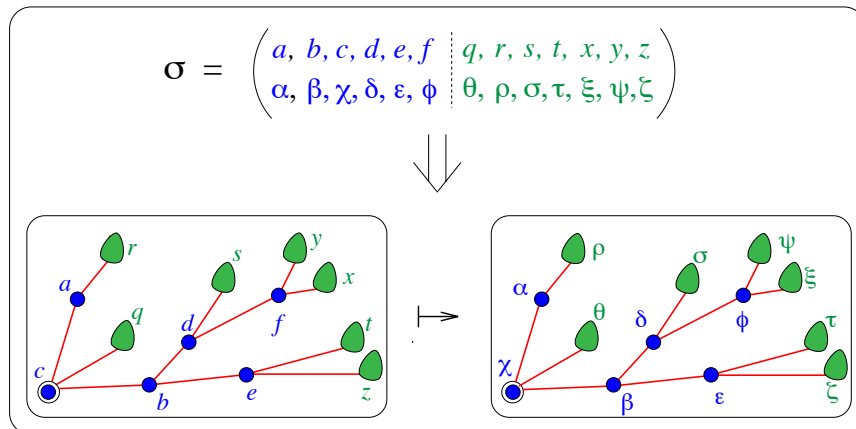

Figure 3.6: Multisort transport preserves colors/sort.

These preliminary considerations justify the definitions which follow.

Definition 3.18. Let $k \geq 1$ be an integer. A **multiset** (with k sorts of elements) is a k-tuple of sets $U = (U_1, \ldots, U_k)$. For brevity we say that U is a k-**set**. An element $u \in U_i$ is called an element of U of sort i. The **multicardinality** of U is the k-tuple of cardinalities $|U| = (|U_1|, \ldots, |U_k|)$. The **total cardinality** of U is the sum

$$\|U\| = |U_1| + \ldots + |U_k|.$$

Definition 3.19. A **multifunction** f from $(U_1, \ldots, U_k)$ to $(V_1, \ldots, V_k)$, denoted by

$$f : (U_1, \ldots, U_k) \longrightarrow (V_1, \ldots, V_k),$$

is a k-tuple of functions $f = (f_1, \ldots, f_k)$ such that $f_i : U_i \longrightarrow V_i$, for $i = 1, \ldots, k$. The composition of two multifunctions is made componentwise. The multifunction f is said to bebijective if each function f_i is bijective.

Definition 3.20. Let $k \geq 1$ be an integer. A **species of k sorts** (or **k-sort species**) is a rule F which produces

  i) for each finite multiset $U = (U_1, \ldots, U_k)$, a finite set $F[U_1, \ldots, U_k]$,

  ii) for each bijective multifunction

$$\sigma = (\sigma_1, \ldots, \sigma_k) : (U_1, \ldots, U_k) \longrightarrow (V_1, \ldots, V_k),$$

   a function

$$F[\sigma] = F[\sigma_1, \ldots, \sigma_k] : F[U_1, \ldots, U_k] \longrightarrow F[V_1, \ldots, V_k].$$

Moreover, the functions $F[\sigma]$ must satisfy the functoriality properties, that is to say, for bijective multifunctions $\sigma = U \longrightarrow V$ and $\tau : V \longrightarrow W$, and for the multifunction identity $\mathrm{Id}_U : U \longrightarrow U$, it is required that

  a) $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$,

  b) $F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$.

As before, an element $s \in F[U_1, \ldots, U_k]$ is called an F-structure on $(U_1, \ldots, U_k)$ (or a structure of species F on $(U_1, \ldots, U_k)$). The function $F[\sigma_1, \ldots, \sigma_k]$ is thetransport of F-structures along $(\sigma_1, \ldots, \sigma_k)$. If $t = F[\sigma_1, \ldots, \sigma_k](s)$, then s and t are said to be isomorphic F-structures. Again, this is an equivalence relation for which the classes are still unlabelled F-structures.

   The usual graphical conventions for the representation of F-structures extend to the multisort context (see Figure 3.7, for $k = 3$) by associating to each element of the underlying set a number (or a shape, or a color, ...) identifying its sort. Of course, it is not necessary that there be at least one element of each sort in an F-structure. Thus, every multisort species can also be viewed as a multisort species on a larger set of sorts.

Remark 3.21. It is often useful, in practice, to represent the multiset $U = (U_1, \ldots, U_k)$ underlying a structure as being the set $U_1 + \ldots + U_k$ (disjoint union $\bigcup_{i=1}^{k} U_i \times \{i\}$), also denoted by U. There follows a function $\chi = \chi_U : U \longrightarrow [k]$, associating to each element its sort, that is to say, the fibers $\chi_U^{-1}(\{i\})$ being the U_i, $i = 1, \ldots, k$. In this case, the multifunctions are identified with ordinary functions $f : U \longrightarrow V$ preserving sorts, that is to say, such that $\chi_V \circ f = \chi_U$.

   We leave to the reader the task of formulating in functorial terms the definition of multisort species and of showing the equivalence of the different points of view presented in the preceding remark (see Exercise 3.19). To each sort, one can associate a species of singletons.
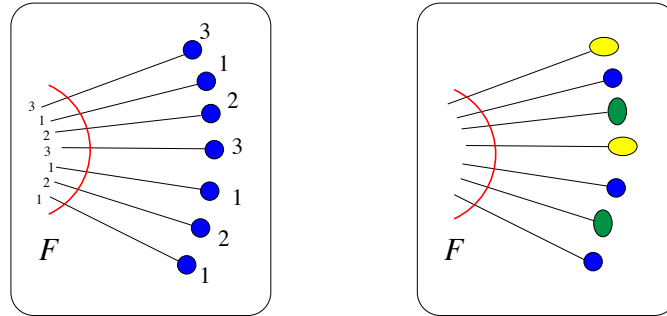
Figure 3.7: Typical multisorted F-structure.

Definition 3.22. For each i, $1 \leq i \leq k$, the (k-sort) species X_i of **singletons of sort** i is defined by

$$X_i[U] = \begin{cases} \{U\}, & \text{if } |U_i| = 1, \text{ and } U_j = \emptyset, \text{ for all } j \neq i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In other words, there is an X_i-structure on $(U_1, \ldots, U_k)$ only if

$$U = \emptyset + \emptyset + \ldots + \underbrace{\{u\}}_{i^{\text{th}} \text{ position}} + \emptyset + \ldots + \emptyset = \{u\}$$

is a singleton of sort i. In this case, $\{u\}$ is the unique X_i-structure.

Remark 3.23. Other variables can be used to designate the species of singletons. The variables currently utilized are the letters of the alphabet X, Y, Z, T, with or without indices. Note the abuse of language which consists in saying, for example, that a Y-structure is a singleton ofsort Y. One often writes $F = F(X_1, \ldots, X_k)$ to indicate that F is a k-sort species. This notation is moreover compatible with the substitution of species defined later in this section.

3.2.2 Weighted multisort species

By combining the notions of multisort species and weighted species, the more general concept of weighted multisort species is obtained.

Definition 3.24. A **weighted multisort species** $F = F_w$ (of k sorts) is a rule which associates

i) to each multiset $U = (U_1, U_2, \ldots, U_k)$, a weighted set $(F[U], w_U)$,

ii) to each bijective multifunction $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ a function $F[\sigma_1, \sigma_2, \ldots, \sigma_k]$, which **preserves the weights**, in such a way that the functoriality conditions of Definition 3.20 are satisfied.

The reader will easily define for himself, following analogous previous contexts, the corresponding notions of isomorphic structures, isomorphism types, equipotent species, and isomorphic species (combinatorial equality). The operations of addition, multiplication, substitution, differentiation, cartesian product, pointing and functorial composition are also defined essentially in the same manner as in the weighted unisort case, in analogy with functions in many variables. This is made easier by the introduction of some terminology for a k-set $U = (U_1, \ldots, U_k)$.

- A k-**set dissection** of U is a pair k-sets (V, W) such that for $i = 1, \ldots, k$, $U_i = V_i \cup W_i$ and $V_i \cap W_i = \emptyset$. We denote by $\Delta[U]$, the set of dissections of U.

- A k-**set partition** π of U is a partition of the total set $U_1 + \ldots + U_k$. Each class $C \in \pi$ can be viewed as a multiset of k sorts, where $C_i = C \cap U_i$. We denote by $\mathrm{Par}[U]$, the set of partitions of U.

Definition 3.25. For two weighted k-sort species F and G and a k-set $U = (U_1, U_2, \ldots, U_k)$, we naturally set $(F + G)[U] = F[U] + G[U]$, and

$$(F{\cdot}G)[U] = \sum_{(V,W)\in\Delta[U]} F[V] \times G[W],$$

The sums and products in the statements above being taken in the sense of weighted sets. Keeping going with our extension of operations to the weighted multisort context, let $F = F(Y_1, \ldots, Y_m)$ be a weighted m-sort species, and $(G_j)_{j=1,\ldots,m}$ be a family of weighted k-sort species. Thepartitional composition $F(G_1, \ldots, G_m)$ (substitution of the G_j, $j = 1..m$ in F) is a k-sort species defined by setting, for $U = (U_1, \ldots, U_k)$,

$$F(G_1, \ldots, G_m)[U] = \sum_{\substack{\pi\in\mathrm{Par}[U] \\ \chi:\pi\to[m]}} F[\chi^{-1}] \times \prod_{\substack{j\in[m] \\ C\in\chi^{-1}(j)}} G_j[C],$$

where, for each function $\chi : \pi \longrightarrow [m]$, χ^{-1} denotes the m-set $(\chi^{-1}(i), \ldots, \chi^{-1}(m))$ associated to χ. In descriptive terms, an $F(G_1, \ldots, G_m)$-structure is an F-structure in which each element of sort Y_j has been inflated into a cell which is a structure of species G_j. By definition, the weight of such a structure s is the product of the weights of the F-structure and the G_j-structures which form s.

Example 3.26. Let $F_{r,s,t}(X, Y) = \mathcal{C}_{\mathrm{alt}}(\mathcal{A}_w(X), \mathcal{G}_v(X + Y))$, where $\mathcal{C}_{\mathrm{alt}}(X, Y) = \mathcal{C}(X{\cdot}Y)$ is the species of alternating oriented cycles of two sorts of elements, weighted by t^n (n being the length of the cycle), $\mathcal{G}_v$ is the species of simple graphs weighted by r^e (where e is the number of edges of the graph), and $\mathcal{A}_w$ is the species of rooted trees weighted by s^f (where f is the number of leaves of the rooted tree). Figure 3.8 represents an $F_{r,s,t}$-structure as well as its weight.

For $F = F_w(Y_1, \ldots, Y_m)$, the functorial composition $F \square (G_1, \ldots, G_m) = F[G_1, \ldots, G_m]$ is only defined if each G_i is not weighted. One then sets

$$F \square (G_1, \ldots, G_m)[U_1, \ldots, U_k] = F[G_1[U_1, \ldots, U_k], \ldots, G_m[U_1, \ldots, U_k]].$$
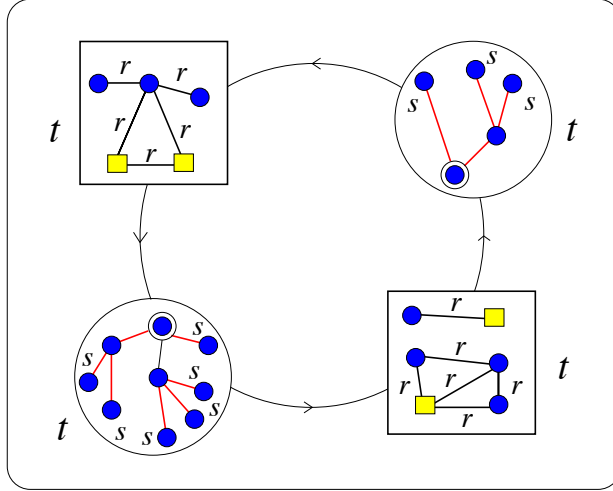
Figure 3.8: An $F_{r,s,t}$-structure of weight $r^{11}s^9t^4$.

The weight of an $F_w[G_1,\ldots,G_m]$-structure s on $(U_1,\ldots,U_k)$ is, by definition, its weight as an F_w-structure on $(G_1[U_1,\ldots,U_k],\ldots,G_m[U_1,\ldots,U_k])$. There are as many notions of partial differentiations $\frac{\partial}{\partial X_i}$ as sorts. For a k-sort species $F = F_w(X_1,\ldots,X_k)$, one sets

$$(\frac{\partial}{\partial X_i}F)[U_1,\ldots,U_k] = F[U_1,\ldots,U_i+\{*_i\},\ldots,U_k],$$

the weight of a $(\partial F/\partial X_i)$-structure s on $(U_1,\ldots,U_k)$ being equal to the weight of s as an F-structure of $(U_1,\ldots,U_i+\{*_i\},\ldots,U_k)$. The usual rules of differential calculus remain valid for multisort species. For example, the following partial differentiation chain rule holds: if F, G and H are two-sort species (X and Y), then

$$\frac{\partial}{\partial X}F(G,H) = F_X(G,H)\frac{\partial G}{\partial X} + F_Y(G,H)\frac{\partial H}{\partial X},$$

where $F_X = \partial F/\partial X$. There are also k operations of pointing which are carried out by setting, for $i = 1,\ldots,k$,

$$F^{\bullet_i} = X_i\frac{\partial}{\partial X_i}F.$$

We leave to the reader the task of formulating the precise definitions of the diverse combinatorial operations introduced, with regard to the transport of structures.

The generating series of weighted multisort species are defined by introducing one formal variable $x,y,z,t,\ldots$ for each sort $X,Y,Z,T,\ldots$. For the case of index series, it is necessary to introduce an infinite number of formal variables

$$x_1,x_2,\ldots;y_1,y_2,\ldots;z_1,z_2,\ldots;t_1,t_2,\ldots;\ldots ,$$

for each sort $X, Y, Z, T, \ldots$. Here is the definition of these series in the case of two sorts. It is straightforward to state the corresponding definitions in the general case.

Definition 3.27. Let $F = F_w(X, Y)$ be a weighted two-sort species. The generating series $F_w(x, y)$, the type generating series $\widetilde{F_w}(x, y)$ and thecycle index series Z_{F_w} are defined by

$$F_w(x, y) = \sum_{n,k \geq 0} |F[n, k]|_w \, \frac{x^n}{n!} \frac{y^k}{k!},$$

where $|F[n, k]|_w$ is the total weight of F-structures on $([n], [k])$,

$$\widetilde{F_w}(x, y) = \sum_{n,k \geq 0} |F[n, k]/\sim|_w \, x^n y^k,$$

where $|F[n, k]/\sim|_w$ is the total weight of unlabelled F-structures on $([n], [k])$,

$$Z_{F_w}(x_1, x_2, x_3, \ldots; y_1, y_2, y_3, \ldots) = \sum_{n,k \geq 0} \frac{1}{n!k!} \sum_{\substack{\sigma \in \mathcal{S}_n \\ \tau \in \mathcal{S}_k}} |\mathrm{Fix}\, F[\sigma, \tau]|_w \, x_1^{\sigma_1} x_2^{\sigma_2} \ldots y_1^{\tau_1} y_2^{\tau_2} \ldots,$$

where $|\mathrm{Fix}\, F[\sigma, \tau]|_w$ is the total weight of F-structures on $([n], [k])$ that are left fixed under transport along (σ, τ).

One has the formulas

$$
\begin{aligned}
F_w(x, y) &= Z_{F_w}(x, 0, 0, \ldots; y, 0, 0, \ldots), & (3.13) \\
\widetilde{F_w}(x, y) &= Z_{F_w}(x, x^2, x^3, \ldots; y, y^2, y^3, \ldots), & (3.14)
\end{aligned}
$$

and the passage to series is compatible with the combinatorial operations $+$, $\cdot$, $\circ$, $'$, $\times$, $\bullet$, and $\square$. Let us describe explicitly the case of substitution. If $F = F_w(X, Y)$, $G = G_u(X, Y)$ and $H = H_v(X, Y)$ are three weighted two-sort species X and Y such that $G(0, 0) = 0 = H(0, 0)$, then

a) $F_w(G_u, H_v)(x, y) = F_w(G_u(x, y), H_v(x, y))$,

b) $\widetilde{F_w(G_u, H_v)}(x, y) = Z_{F_w}(\widetilde{G_u}(x, y), \widetilde{G_{u^2}}(x^2, y^2), \ldots; \widetilde{H_v}(x, y), \widetilde{H_{v^2}}(x^2, y^2), \ldots)$,

b) $Z_{F_w(G_u, H_v)} = Z_{F_w}(Z_{G_u}, Z_{H_v}) = Z_{F_w}((Z_{G_u})_1, (Z_{G_u})_2, \ldots; (Z_{H_v})_1, (Z_{H_v})_2, \ldots)$,

where, for $k \geq 1$, $(Z_{G_u})_k(x_1, x_2, x_3, \ldots) = Z_{G_{u^k}}(x_k, x_{2k}, x_{3k}, \ldots)$. This last substitution is the plethystic composition of weighted index series, in the variables x_i and y_i, $i = 1, 2, 3, \ldots$, defined in an fashion analogous to plethystic substitution for weighted unisort species (see Definition 3.9). These properties constitute, once more, powerful computational tools in concrete applications.

Example 3.28. Consider three sorts of elements X, Y and Z. Their sum forms the species $X + Y + Z$. An $(X + Y + Z)$-structure is then a singleton of one of the three sorts X, Y or Z. Substitution into a unisort species $F(X)$ gives the species $F(X+Y+Z)$. An $F(X+Y+Z)$-structure is an F-structure placed on a finite (multi-) set formed from three sorts of elements (possessing an arbitrary number, possibly zero, of elements of each sort). Thus, the species $\mathcal{S}^{\text{tric}}$ of tri-colored permutations can be written in the form $\mathcal{S}^{\text{tric}}(X, Y, Z) = \mathcal{S}(X + Y + Z)$, $\mathcal{S} = \mathcal{S}(X)$ being the usual species of permutations on a single sort X. More generally, one can substitute any sum of sorts (weighted or not, with or without repetitions) for each variable in a weighted multisort species. For example, beginning with a weighted multisort species $F_w(X, Y, Z, T, \ldots)$, one can form the species $F_w(3X + T, X_s, X_s, 5X + 4Y_t, \ldots)$, where X_s (respectively, Y_t) denotes the species of singletons of sort X and of weight s (respectively, of sort Y and of weight t). It is interesting to note the following combinatorial equations:

$$
\begin{aligned}
F_w(mX_s, nY_t, \ldots) &= F_w(X, Y, \ldots) \times E(mX_s + nY_t + \ldots) \\
&= F_w(X, Y, \ldots) \times (E^m(X_s) \cdot E^n(Y_t) \cdot \ldots).
\end{aligned}
$$

The example of tri-colored permutations $\mathcal{S}^{\text{tric}}(X, Y, Z) = \mathcal{S}(X + Y + Z)$ could lead one to ask if for any multisort species $F = F(X, Y, Z, \ldots)$ there exists a species with one sort $H = H(X)$ such that $F(X, Y, Z, \ldots)$ is of the form $H(X + Y + Z + \ldots)$. This is false (just as for functions in many variables).

Example 3.29. For a combinatorial verification, it suffices, for example, to examine the species $\mathcal{C}_{\text{alt}}(X, Y)$ of alternating oriented cycles on two sorts. To obtain a $\mathcal{C}_{\text{alt}}$-structure, one must alternately place elements of sort X and of sort Y to form an oriented cycle (see Figure 3.9 a)). We
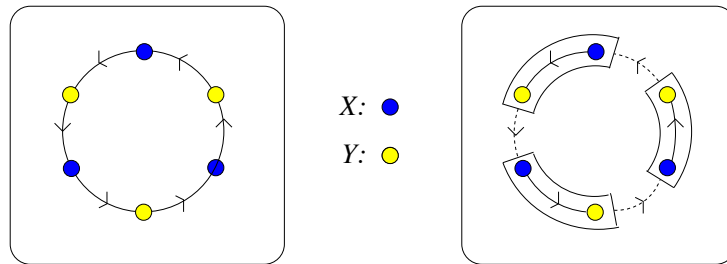

Figure 3.9: a) Alternating color cycle.   b) Cycle of colored pairs.

are going to show that for any unisort species $H = H(X)$, we have $\mathcal{C}_{\text{alt}}(X, Y) \neq H(X + Y)$. Indeed, making the substitution $Y := 0$ in the equation gives $\mathcal{C}_{\text{alt}}(X, 0) = H(X)$. An H-structure must then be an alternating cycle not having any element of sort Y. But an alternating cycle must have as many elements of each sort, so we deduce that such a structure cannot exist (since a cycle must always contain at least one element). Thus $H = 0$ (the empty species) and we deduce $\mathcal{C}_{\text{alt}}(X, Y) = 0$, which is false. Figure 3.9 b) shows, however, that one has the combinatorial equation $\mathcal{C}_{\text{alt}}(X, Y) = \mathcal{C}(X \cdot Y)$, where $\mathcal{C} = \mathcal{C}(X)$ is the species of oriented cycles (on the sort X). The

following series are immediately deduced:

$$\mathcal{C}_{\text{alt}}(x,y) = \ln\left(\frac{1}{1-xy}\right), \quad \widetilde{\mathcal{C}_{\text{alt}}}(x,y) = \frac{xy}{1-xy}, \quad \text{and} \quad Z_{\mathcal{C}_{\text{alt}}} = \sum_{k \geq 1} \frac{\varphi(k)}{k} \ln\left(\frac{1}{1-x_k y_k}\right).$$

Another interesting example of a combinatorial equation comes from the following decomposition of the species $\mathcal{C}(X+Y)$ of oriented cycles on two sorts X, Y of elements:

$$\mathcal{C}(X+Y) = \mathcal{C}(X) + \mathcal{C}(YL(X)).$$

This formula expresses the fact that if a $\mathcal{C}(X+Y)$-structure has at least one element of sort Y, then it can be identified, in a natural manner, to an oriented cycle formed from disjoint chains of the form $yx_1x_2\ldots x_k$, where y is an element of sort Y and the x_i are distinct elements of sort X (see Figure 3.10). The reader is invited to analyze the precise form which this equation takes when passing to generating and index series (see Exercise 3.24). Every weighted multisort species
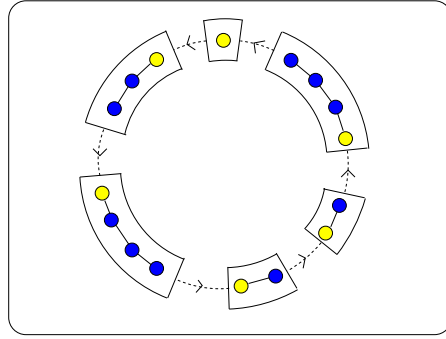

Figure 3.10: Cycle of disjoint chains.

$F_w = F_w(X, Y, Z, \ldots)$ has a canonical decomposition of the form

$$F_w = \sum_{\mathbf{n} \geq 0} F_{w;\mathbf{n}},$$

where $F_{w;\mathbf{n}}$ denotes the multisort species F_w restricted to the multicardinality $\mathbf{n} = (m, n, p, \ldots)$, defined by

$$F_{w;\mathbf{n}}[U, V, W, \ldots] = \begin{cases} F_w[U, V, W, \ldots], & \text{if } (|U|, |V|, |W|, \ldots) = \mathbf{n}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

To end the present section, we examine an important operation that can be performed on multisort species: the passage to isomorphism types according to one of the sorts, illustrated here in the case of two-sort species.

Definition 3.30. Let $F = F_v(X, Y)$ be an $\mathcal{R}$-weighted two-sort species. Consider two F-structures $s \in F[U, V]$ and $t \in F[U', V']$. One says that s and t have the same **isomorphism type according to the sort** Y (and one writes $s \sim_Y t$) if $U = U'$, and t is obtained from s by transport of structures along a bijection of the form $\sigma = Id + \theta : U + V \longrightarrow U + V'$, where $\theta : V \longrightarrow V'$ is a bijection.

The class of s according to the equivalence relation $\sim_Y$ is called the **type of s according to Y** and is denoted by $\mathrm{T}_Y s$. One says that U is the underlying set of $\mathrm{T}_Y s$. In more visual terms, one has $s \sim_Y t$ if and only if s and t become equal when the elements of sort Y are made indistinguishable in their underlying sets. The structure thus obtained represents the type $\mathrm{T}_Y s$. For each finite set U, we can consider the set of types according to Y $\mathrm{T}_Y F[U] = \{\mathrm{T}_Y s \mid \exists V, s \in F[U,V]\}$, of which U is the underlying set. This set is infinite in general, but summable in $\mathcal{R}[\![y]\!]$ if we introduce a variable y as a counter for the points of sort Y, by analogy with the isomorphism types series of unisort species. In other words, we define the weight of a type $\mathrm{T}_Y s$, for $s \in F_v[U,V]$, by setting $w(\mathrm{T}_Y s) = v(s)\, y^{|V|}$. We then have

$$|\mathrm{T}_Y F[U]|_w = \widetilde{F_v[U,Y)}(y),$$

where $F_v[U,Y)$ denotes the species of one sort Y derived from $F_v(X,Y)$ by keeping the first component fixed $(= U)$. This gives a species of $\mathcal{R}[\![y]\!]$-weighted structures, denoted $\tau_{Y;y}F_v$, called **the species of types of $F(X,Y)$-structures according to the sort Y**. The definition of transport functions is left to the reader. It is sometimes possible to set $y = 1$ in this process. A sufficient condition is that the sets $\mathrm{T}_Y F[U]$ be finite, or that the species $F(X,Y)$ be polynomial in Y, in the following sense.

Definition 3.31. Let $F = F(X,Y)$ be a species on two sorts X, Y, whose canonical decomposition is

$$F = \sum_{n,k \geq 0} F_{n,k}.$$

We say that F is **polynomial in Y** if for any $n \geq 0$, there exists $N \geq 0$ such that $k \geq N$ implies $F_{n,k} = 0$.

When the species $F = F_v(X,Y)$ is polynomial in Y, then for each finite set U the set $\mathrm{T}_Y F[U]$ of types according to Y is a finite union of sets, that are finite or summable in $\mathcal{R}$, of the form $F[U,V]/\sim_Y$. It is then itself finite or summable in $\mathcal{R}$ and thus determines a species of structures, denoted by $\mathrm{T}_Y F$, for which $\mathrm{T}_Y F = \tau_{Y;y} F\big|_{y=1}$, and which could also be denoted by $F(X,1)$ (see [163], Section 2.1). More generally, one can define (see Exercise 3.28) the types of a species $F_w(X_1, X_2, \ldots, X_k)$ according to the sort X_i as well as the $(k-1)$-sort species

$$\tau_{X_i;x_i} F_w = \tau_{X_i;x_i} F_w(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k).$$

Here is an example of a species of the form $\mathrm{T}_Y F$.

Example 3.32. Consider the species $\Gamma = \Gamma(X,Y)$ of graphs constructed on vertices (of sort X) and edges (of sort Y). A Γ-structure on a pair (U,V) is then a graph in which the set of vertices is U and the edges (all distinguishable from one another) form the set V. We see also in this case that Γ is polynomial in Y, for we can select $N = \binom{n}{2} + 1$ in the preceding definition. Figure 3.11 a) illustrates a Γ-structure on the set $U = \{a,b,c,d,e,f\}$ of **vertices** and the set $V = \{m,n,p,q,r,s\}$ of **edges**, whereas Figure 3.11 b) shows the $\mathrm{T}_Y \Gamma$-structure to which it corresponds, of weight y^6.

This $\mathrm{T}_Y\Gamma$-structure is quite simply a graph in the usual sense since the edges (represented by line segments) have become indistinguishable. We have the equation $\mathcal{G}_v = \mathrm{T}_{Y;y}\Gamma$, where $\mathcal{G}_v = \mathcal{G}_v(X)$ denotes the species of simple graphs, with an edge counter y, that is with weight $v(g) = y^{e(g)}$, where $e(g)$ is the number of edges of the graph g. Moreover, by considering multigraphs, where multiple edges are allowed, we obtain a weighted species admitting a (summable) infinite set of structures (see Exercise 3.17).
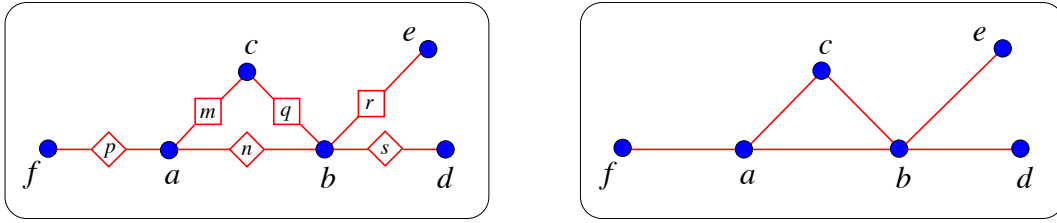

Figure 3.11: a) b).

Note the following formulas for the series associated to species of the form $\mathrm{T}_{Y;y}F$:

a) $Z_{\mathrm{T}_{Y;y}F}(x_1, x_2, x_3, \ldots) = Z_F(x_1, x_2, x_3, \ldots; y, y^2, y^3, \ldots),$

b) $(\widetilde{\mathrm{T}_{Y;y}F})(x) = Z_F(x, x^2, x^3, \ldots; y, y^2, y^3, \ldots),$ (3.15)

c) $\mathrm{T}_{Y;y}F(x) = Z_F(x, 0, 0, \ldots; y, y^2, y^3, \ldots),$

where $Z_F(x_1, x_2, x_3, \ldots; y_1, y_2, y_3, \ldots)$ denotes the cycle index series of the weighted two-sort species $F = F_v(X, Y)$. (See Exercise 3.20 d)).

3.3 Exercises

Exercises for Section 3.1

Exercise 3.1. Let $\mathbb{K} \subseteq \mathbb{C}$ be an integral domain and $\mathcal{R} = \mathbb{K}[\![t_1, t_2, \ldots]\!]$, a ring of formal power series in the variables $t_1, t_2, \ldots$. We say that the weighted set (A, w), where $w : A \to \mathcal{R}$, is summable if, for any monomial $\mu = t_1^{n_1} t_2^{n_2} \ldots$ in the variables $t_1, t_2, \ldots$, the following set

$$W_\mu = \{a \in A \mid [\mu]\, w(a) \neq 0\}$$

is finite (recall that $[\mu]w(a) = [t_1^{n_1} t_2^{n_2} \ldots]\, w(a)$ denotes the coefficient of the monomial μ in the formal series $w(a)$). We define the inventory (or total weight or cardinal) of a summable weighted set (A, w), as being the unique element of $\mathcal{R}$ (i.e., the formal series), denoted by $|A|_w = \sum_{a \in A} w(a)$, satisfying

$$[\mu]\, |A|_w = \sum_{a \in W_\mu} [\mu]\, w(a)$$

for any monomial $\mu = t_1^{n_1} t_2^{n_2} \dots$. Let (A, w) be a weighted summable set.

a) Show that if $S \subseteq A$ and $w_S : S \to \mathcal{R}$ denotes the restriction of w to S, then the weighted set (S, w_S) is summable.

b) Show that if (B, v) is a weighted set such that $(A, w) \simeq (B, v)$ (see Definition 3), then (B, v) is summable and $|A|_w = |B|_v$.

c) Show that if (B, v) is a summable weighted set then the weighted sets sum, $(A, w) + (B, v)$, and product, $(A, w) \times (B, v)$ (see Definition 4), are summable and the formulas (3.2) are valid.

Exercise 3.2.   a) Let $A = \mathbb{N}_+ = \{1, 2, 3, \dots\}$ and set $w(a) = t_1 t_2^2 \dots t_a^a$ for all $a \in A$. Show that A is summable and calculate the inventory $|A|_w$.

b) Denote by $\tau(n)$, the number of divisors of an integer $n \geq 1$. Is the weighted set $(\mathbb{N}_+, v)$, where $v(n) = t^{\tau(n)}$, summable?

c) Let A be the (infinite) set of all the trees on the set of vertices $\{1, 2, \dots, n\}$, where n runs over $\mathbb{N}_+$. Define the weight $w(a)$ of a tree a by $w(a) = t_1^{n_1} t_2^{n_2} \dots$, where n_i is the number of vertices of degree i in $a, i = 1, 2, 3, \dots$. Show that A is summable and interpret the inventory $|A|_w$ combinatorially.

Exercise 3.3. Let A, B and C be disjoint sets, with $|A| = a$, $|B| = b$, and $|C| = c$. Denote by $\mathrm{Inj}(A, A \cup B)$ the set of injective functions $f : A \to A \cup B$ weighted by $w(f) = \lambda^{\mathrm{cyc}(f)}$.

a) Show that

$$|\mathrm{Inj}(A, A \cup B)|_w = (\lambda + b)^{<a>} = (\lambda + b) \ (\lambda + b + 1) \dots (\lambda + b + a - 1)$$

b) Establish an isomorphism of weighted sets

$$\mathrm{Inj}(A \cup B, A \cup B \cup C) \xrightarrow{\sim} \mathrm{Inj}(A, A \cup C) \times \mathrm{Inj}(B, A \cup B \cup C)$$

and write the identity which corresponds to it.

Exercise 3.4.   a) Establish Proposition 11, excluding (3.4), e′).

b) Show that Definition 9 makes sense and verify the summability conditions which are implicit in formula (3.3).

Exercise 3.5.   a) Verify formulas (3.5) giving the generating and index series of the species $\mathcal{S}_w$ of permutations with cycle counter α.

b) Show moreover that

i) $|\mathcal{S}[n]|_w = \alpha^{<n>} = \alpha(\alpha + 1) \dots (\alpha + n - 1)$,

ii) for a permutation σ of type $(\sigma_1, \sigma_2, \dots)$

$$|\text{Fix}\,\mathcal{S}[\sigma]|_\omega = \prod_{j \geq 1} \omega_j(\alpha)\,(\omega_j(\alpha) + j) \cdots (\omega_j(\alpha) + j\,(\sigma_j - 1)).$$

where $\omega_n(\alpha) = \sum_{d|n} \phi(d)\alpha^{n/d}$.

c) Verify formulas (3.6) giving the generating and index series of the species Par_w of weighted partitions by block count.

d) Show that

$$\sum_{n \geq 0} \sum_{k=0}^{n} S(n,k) t^k \frac{x^n}{n!} = \exp(t(e^x - 1)).$$

Exercise 3.6.   a) Prove formulas (3.7) giving the generating and index series of the weighted species End_v of endofunctions, described in Example 3.14.

b) Establish formula (3.8) for $|\text{End}[n]|_v$. **Hint:** Take as known that the number of forests made up of k rooted trees on n vertices is given by $\frac{k}{n}\binom{n}{k}n^{n-k}$, $n > 0$.

Exercise 3.7. LYNDON WORDS AND TYPES OF COLORED PERMUTATIONS. Consider the **alphabet**

$$A = \mathbb{N}_+ = \{1, 2, 3, \dots\},$$

weighted by the function $w(i) = t_i$. Then $|A|_w = p_1$, and more generally, for $k \geq 1$, $|A|_{w^k} = t_1^k + t_2^k + \cdots = p_k$. Denote by A^n the set of all **words** of length n ($n \geq 0$) and by A^* the set of all the words in the alphabet A. One extends the weighting w to A^* by associating to the word $ij\ldots k \in A^*$, the commutative word (or monomial) $t_i t_j \ldots t_k$. A^* forms a monoid under concatenation. Two words m and m' are said to be **conjugates** if there exists a factorization $m = uv$ so that $m' = vu$. This constitutes an equivalence relation for which the classes are called the **circular words**. A **Lyndon word** is a primitive (i.e., not a positive power of another word) word which is smaller than all its conjugates with respect to the lexicographic order. Denote by $C(n)$ the set of circular words of length n and by $L(n)$, the set of Lyndon words of length n.

a) Show that for $n \geq 1$,

i) $|C(n)|_w = \sum_{d|n} |L(n/d)|_{w^d}$,

ii) $|A^n|_w = p_1^n = \sum_{d|n} \frac{n}{d}\,|L(n/d)|_{w^d}$,

iii) $|L(n)|_w = \frac{1}{n}\sum_{d|n} \mu(d)\,p_d^{n/d}$,

iv) $|C(n)|_w = \frac{1}{n}\sum_{d|n} \phi(d)\,p_d^{n/d}$,

Hint: Here μ denotes the Möbius function and ϕ Euler's ϕ-function. Use the fact that $\phi(n) = n\sum_{d|n} \frac{\mu(d)}{d}$.

b) Consider the weighted species $X_\tau = X_{t_1} + X_{t_2} + \ldots$, the species $\mathcal{C}$ of cyclic permutations, and the composite species $\mathcal{C}(X_\tau)$, whose structures are called **colored cycles**. Show that there exists an isomorphism of weighted sets between $(C(n), w)$ and unlabeled $\mathcal{C}_n(X_\tau)$−structures and that

$$\widetilde{\mathcal{C}(X_\tau)}(x) = \sum_{n \geq 1} \frac{x^n}{n} \sum_{d|n} \phi(d)\, p_d^{n/d} = \sum_{m \geq 1} \frac{\phi(m)}{m} \log \frac{1}{1 - p_m x^m}.$$

Deduce formula (2.20) giving the index series $Z_\mathcal{C}$.

c) A colored cycle is called **asymmetric** if its only automorphism is the identity. Denote by $\overline{\mathcal{C}(X_\tau)}$, the species of colored asymmetric cycles. Show that there exists an isomorphism of weighted sets between Lyndon words and unlabeled colored asymmetric cycles. Deduce that

$$\overline{\mathcal{C}(X_\tau)}(x) = \sum_{n \geq 1} \frac{x^n}{n} \sum_{d|n} \mu(d)\, p_d^{n/d} = \sum_{m \geq 1} \frac{\mu(m)}{m} \log \frac{1}{1 - p_m x^m}.$$

d) Show that every word $\mu \in A^*$ can be written in a unique manner as a product $\mu = \ell_1 \ell_2 \ldots \ell_k$, where the ℓ_i are Lyndon words and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$ with respect to the lexicographic order (see Chapter 5 of [227] or Chapter 8 of [280]). Deduce that the series $|A^*|_w = 1/(1 - p_1)$ can be viewed as the inventory of unlabeled assemblies of asymmetric colored cycles.

e) Show that $|L(n)|_{w^k}$ is the inventory of words of length nk that are circular and k-**symmetric**, that is to say, of the form u^k, where u is primitive. Show that the series $1/(1 - p_k)$ can be considered as the inventory of unlabeled assemblies of k-symmetric colored cycles, that is to say, whose automorphism group is the cyclic group of order k.

f) An $\mathcal{S}(X_\tau)$-structure is called a **colored permutation**. It is an assembly of colored cycles. By regrouping these colored cycles according to the order of their automorphism group, deduce from e) that

$$\widetilde{\mathcal{S}(X_\tau)}(x) = \prod_{k \geq 1} \frac{1}{1 - p_k\, x^k}$$

g) A colored permutation is called asymmetric if its only automorphism is the identity. It consists of an injective assembly (i.e., whose members are pairwise non-isomorphic) of colored asymmetric cycles. Denote by $\overline{\mathcal{S}(X_\tau)}$, the species of asymmetric colored permutations. Show that the series $p_2/(1-p_1)$ can be viewed as the inventory of unlabeled non-injective assemblies of asymmetric colored cycles (see [266]). Deduce that

$$\overline{\mathcal{S}(X_\tau)}(x) = \frac{1 - p_2 x^2}{1 - p_1 x}$$

Exercise 3.8. HERMITE POLYNOMIALS. The (unitary) Hermite polynomials can be defined by formulas (3.10) or (3.11), that is to say from the combinatorial model of involutions φ weighted by $w(\varphi) = t^{\varphi_1}(-1)^{\varphi_2}$

a) Show that $H_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-2)^{-k} \dfrac{n!}{k!(n-2k)!} t^{n-2k}$.

b) A variant $\overline{H}_n(t)$ of the Hermite polynomials is defined by the renormalization $\overline{H}_n(t) = 2^{n/2} H_n(t\sqrt{2})$. Show that

$$\sum_{n \geq 0} \overline{H}_n(t) \, \frac{x^n}{n!} = \exp(2tx - x^2).$$

c) (Chapter 5 of [60]) Establish formulas (3.12) and show that for any permutation σ of type $(\sigma_1, \sigma_2, \dots)$

$$|\text{Fix Inv}[\sigma]|_w = \prod_{j \geq 1} ((-1)^{j-1} j)^{\frac{\sigma_j}{2}} H_{\sigma_j}(\xi_j),$$

where

$$\xi_j = \frac{t^j + \chi(j \text{ is even})(-1)^{j/2}}{((-1)^{j-1} j)^{1/2}}.$$

d) Show combinatorially that the polynomial $y = H_n(t)$ satisfies the differential equation $y'' - t\,y' + n\,y = 0$.

   Hint: study the effect of the derivative on the weighted involutions.

e) Show combinatorially that $H_{n+1}(t) = t\,H_n(t) - n\,H_{n-1}(t)$.

f) MEHLER'S FORMULA (SEE [107]). Show that

$$\sum_{n \geq 0} H_n(t_1) H_n(t_2) \frac{x^n}{n!} = \frac{1}{\sqrt{1-x^2}} \exp \frac{t_1 t_2 x - (t_1^2 + t_2^2)\, x^2/2!}{1-x^2},$$

   by considering an appropriate combinatorial model.

g) LINEARIZATION COEFFICIENTS Let $\nu : \mathbb{R}[t] \longrightarrow \mathbb{R}$ be the linear functional defined by $\nu(1) = 1$, $\nu(t^{2n}) = (2n-1)(2n-3)\dots 3 \cdot 1$, for $n \geq 1$, and $\nu(t^{2n+1}) = 0$, if $n \geq 0$. One can show that in fact

$$\nu(t^n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^n e^{-t^2/2} dt.$$

   Denote by $I(n)$, the set of involutions without fixed points. Moreover, for a multicardinal $(n_1, n_2, \dots, n_k)$, let $U_1, U_2, \dots, U_k$ be disjoint sets such that $|U_j| = n_j$, $j = 1, 2, \dots, k$, and denote by $I(n_1, n_2, \dots, n_k)$, the set of involutions without any fixed color on $U = U_1 + U_2 + \dots + U_k$, that is to say, the involutions φ on U such that $u \in U_j \implies \varphi(u) \notin U_j$. Show combinatorially that

   i) $\nu(t^n) = |I(n)|$, for $n \geq 1$,

   ii) $\nu(H_n(t)) = 0$, for $n \geq 1$,

iii) $\nu(H_n(t)H_m(t)) = \begin{cases} n!, & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases}$ (orthogonality)

iv) $\nu(H_{n_1}(t)H_{n_2}(t)\ldots H_{n_k}(t)) = |I(n_1, n_2, \ldots, n_k)|.$

Exercise 3.9. LAGUERRE POLYNOMIALS (SEE FOATA AND STREHL [113]). The Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x)$ can be defined by the generating series

$$\sum_{n \geq 0} \mathcal{L}_n^{(\alpha)}(x) \frac{u^n}{n!} = \left(\frac{1}{1-u}\right)^{\alpha+1} \exp\left(\frac{-xu}{1-u}\right).$$

a) Deduce a combinatorial model with the help of the weighted species $\mathrm{Lag}^{(\alpha)} = E(\mathcal{C}_{\alpha+1}) \cdot E(L_{+(-x)})$, for which we have $\mathcal{L}_n^{(\alpha)}(x) = |\mathrm{Lag}^{(\alpha)}[n]|_w$. The structures of this species are called **Laguerre configurations**.

b) Show that Laguerre configurations can be considered as injective partial endofunctions and that (see Exercise 3.3)

$$\mathcal{L}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}(\alpha + 1 + n - k)^{<k>}(-x)^{n-k}.$$

c) RECURRENCE FORMULA. Show, by a study of these configurations, that

$$\mathcal{L}_{n+1}^{(\alpha)}(x) = (\alpha + 2n - x + 1)\,\mathcal{L}_n^{(\alpha)}(x) - n\,(n + \alpha)\mathcal{L}_{n-1}^{(\alpha)}(x).$$

d) Give a combinatorial interpretation of the derivative $y' = dy/dx$ of the polynomial $y = \mathcal{L}_n^{(\alpha)}(x)$ and prove the following formulas using this interpretation:

i) $xy'' + (\alpha + 1 - x)\,y' + ny = 0,$

ii) $x\dfrac{d}{dx}\mathcal{L}_n^{(\alpha)}(x) = n\,\mathcal{L}_n^{(\alpha)}(x) - n\,(n + \alpha)\,\mathcal{L}_{n-1}^{(\alpha)}(x)$, (F.G. Tricomi)

iii) $\dfrac{d}{dx}\mathcal{L}_n^{(\alpha)}(x) = -n\,\mathcal{L}_{n-1}^{(\alpha+1)}(x).$

e) Let Ψ be the linear functional defined on the polynomials by $\Psi(x^n) = (\alpha+1)^{<n>}$, for $n \geq 0$. It is classical that, for a real $\alpha > -1$,

$$\Psi(x^n) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty x^{n+\alpha}e^{-x}dx.$$

For $(n_1, n_2, \ldots, n_m)$ a multicardinal, let $U_1, U_2, \ldots, U_m$ be disjoint sets such that $|U_j| = n_j$, $j = 1, 2, \ldots, m$, and let $U = U_1 + U_2 + \cdots + U_m$. Denote by $L(n_1, n_2, \ldots, n_m)$, the set of **colored derangements** of U, that is to say, the permutations σ of U such that $u \in U_j \Rightarrow \sigma(u) \notin U_j$, weighted by $w(\sigma) = (-1)^{|U|}(\alpha+1)^{\mathrm{cyc}(\sigma)}$. Show combinatorially that

i) $\Psi(x^n) = \displaystyle\sum_{\sigma \in S[n]} (\alpha + 1)^{\mathrm{cyc}(\sigma)}$,

ii) $\Psi\big(\mathcal{L}_n^{(\alpha)}(x)\big) = 0$, for $n > 0$,

iii) $\Psi\big(\mathcal{L}_n^{(\alpha)}(x)\, \mathcal{L}_m^{(\alpha)}(x)\big) = \begin{cases} n!, \ (\alpha+1)^{<n>} & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$, (orthogonality)

iv) $\Psi(\mathcal{L}_{n_1}^{(\alpha)}(x)\mathcal{L}_{n_2}^{(\alpha)}(x)\ldots\mathcal{L}_{n_m}^{(\alpha)}(x)) = |L(n_1, n_2, \ldots, n_m)|_w$.

Exercise 3.10. WEIGHTED EXPONENTIAL FORMULAS. Consider a weighted species $F = F_w$ and suppose that $F_w = E(F_w^c)$, where F_w^c is the species of connected F_w-structures.

a) Show that

   i) $F_w(x) = \exp F_w^c(x)$,

   ii) $\widetilde{F_w}(x) = \exp \displaystyle\sum_{k \geq 1} \frac{1}{k} \widetilde{F_{w^k}^c}(x^k)$,

   iii) $Z_{F_w}(x_1, x_2, x_3, \ldots) = \exp \displaystyle\sum_{k \geq 1} \frac{1}{k} Z_{F_{w^k}^c}(x_k, x_{2k}, x_{3k}, \ldots)$.

b) Prove the inverse of the preceding formulas

   i) $F_w^c(x) = \log F_w(x)$,

   ii) $\widetilde{F_w^c}(x) = \displaystyle\sum_{k \geq 1} \frac{\mu(k)}{k} \log \widetilde{F_{w^k}}(x^k)$,

   iii) $Z_{F_w^c}(x_1, x_2, x_3, \ldots) = \displaystyle\sum_{k \geq 1} \frac{\mu(k)}{k} \log Z_{F_{w^k}}(x_k, x_{2k}, x_{3k}, \ldots))$.

   where μ denotes the usual Möbius function.

c) Given a formal variable α, define the weighted species $F_{w^{(\alpha)}}$ by setting, for any F_w-structure s, $w^{(\alpha)}(s) = w(s) \cdot \alpha^{c(s)}$, where $c(s)$ is the number of connected components of s. Show that

   i) $F_{w^{(\alpha)}}(x) = (F_w(x))^\alpha$,

   ii) $\widetilde{F_{w^{(\alpha)}}}(x) = \displaystyle\prod_{k \geq 1} \left(\widetilde{F_{w^k}}(x^k)\right)^{\lambda_k(\alpha)}$,

   iii) $Z_{F_{w^{(\alpha)}}}(x_1, x_2, x_3, \ldots) = \displaystyle\prod_{k \geq 1} Z_{F_{w^k}}(x_k, x_{2k}, x_{3k}, \ldots)^{\lambda_k(\alpha)}$,

   where

$$\lambda_n(\alpha) = \frac{1}{n} \sum_{d \mid n} \mu(n/d)\alpha^d.$$

d) Given two formal variables α and β, prove that $F_{w^{(\alpha+\beta)}}(x) = F_{w^{(\alpha)}}(x)F_{w^{(\beta)}}(x)$, but that in general

$$\widetilde{F_{w^{(\alpha+\beta)}}}(x) \neq \widetilde{F_{w^{(\alpha)}}}(x)\widetilde{F_{w^{(\beta)}}}(x),$$
$$Z_{F_{w^{(\alpha+\beta)}}} \neq Z_{F_{w^{(\alpha)}}}Z_{F_{w^{(\beta)}}},$$
$$F_{w^{(\alpha+\beta)}} \neq F_{w^{(\alpha)}}F_{w^{(\beta)}},$$

Hint: Use the fact that for $n \geq 2$, $\lambda_n(\alpha + \beta) \neq \lambda_n(\alpha) + \lambda_n(\beta)$.

e) Given two formal variables α and β, prove the formulas

i) $F_{w^{(\alpha\beta)}}(x) = (F_{w^{(\alpha)}}(x))^{\beta}$,

ii) $\widetilde{F_{w^{(\alpha\beta)}}}(x) = \prod_{k \geq 1} \left(\widetilde{F_{w^k{(\alpha^k)}}}(x^k)\right)^{\lambda_k(\beta)}$,

iii) $Z_{F_{w^{(\alpha\beta)}}}(x_1, x_2, x_3, \ldots) = \prod_{k \geq 1} Z_{F_{w^k{(\alpha^k)}}}(x_k, x_{2k}, x_{3k}, \ldots)^{\lambda_k(\beta)}$.

Hint: Show at first that $w^{(\alpha\beta)} = F_{w^{(\alpha)}(\beta)}$. Deduce that for any $n \geq 1$,

$$\lambda_n(\alpha\beta) = \sum_{dk=n} \lambda_d(\alpha^k)\lambda_k(\beta). \tag{3.16}$$

f) Let $\alpha = |A|$, where A is a finite alphabet.

i) Show that $\lambda_n(\alpha)$ is the number of Lyndon words of length n over A (see Exercise 3.7).

ii) Interpret combinatorially formula (3.16) for $\lambda_n(\alpha\beta)$ in the context of Lyndon words.

Exercise 3.11. a) Let $\nu_n(\alpha) = \frac{1}{n}\sum_{d|n} \phi(d)\alpha^{n/d}$. Show that $\nu_n(\alpha) = \sum_{d|n} \lambda_d(\alpha)$ deduce from Exercise 3.10 c) both formulas (3.5) and (3.6) giving the series associated with the species $\mathcal{S}_w$ and Par_w.

b) Also deduce from Exercise 3.10 c) the **cyclotomic identities** (see [250], and [17],):

i) $\dfrac{1}{1 - \alpha x} = \prod_{k \geq 1} \left(\dfrac{1}{1 - x^k}\right)^{\lambda_k(\alpha)}$,

ii) $\exp(\alpha x_1 + \frac{1}{2}\alpha^2 x_2 + \ldots) = \prod_{k \geq 1} \exp(\lambda_k(\alpha)(x_k + \frac{1}{2}x_{2k} + \ldots))$.

Hint: Consider the species $E(X_\alpha)$, where X_α is the species of singletons of weight α.

Exercise 3.12.   a) Returning to the notation of Exercise 3.10, show that

$$\lambda_n(-1) = \frac{1}{n} \sum_{d|n} \mu(n/d)(-1)^d = \begin{cases} -1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Show first that $-\lambda_n(-1)$ is a multiplicative function in n, using the fact that the convolution

$$(f * g)(n) = \sum_{d|n} f(n/d)g(d)$$

of two multiplicative functions f and g is a multiplicative function (see [3]).

b) Deduce the formulas

i) $F_{w^{(-1)}}(x) = \dfrac{1}{F_w(x)}$,

ii) $\widetilde{F_{w^{(-1)}}}(x) = \dfrac{\widetilde{F_{w^2}}(x^2)}{\widetilde{F_w}(x)}$,

iii) $Z_{F_{w^{(-1)}}}(x_1, x_2, \dots) = \dfrac{Z_{F_{w^2}}(x_2, x_4, \dots)}{Z_{F_w}(x_1, x_2, \dots)}$.

c) In the case where $F_w = \mathcal{S}$, the species of permutations, let $\mathcal{S}_{(-1)} = F_{w^{(-1)}}$, that is, the species of permutations weighted by $(-1)^{\mathrm{cyc}(\sigma)}$. Show that

i) $\mathcal{S}_{(-1)}(x) = 1 - x$,

ii) $\widetilde{\mathcal{S}_{(-1)}}(x) = (1 - x)(1 - x^3)(1 - x^5) \cdots$,

iii) $Z_{\mathcal{S}_{(-1)}}(x_1, x_2, x_3, \dots) = (1 - x_1)(1 - x_3)(1 - x_5) \cdots$.

Interpret these formulas combinatorially.

d) Consider the case where $F_w = E(X_y)$.

Exercise 3.13. Let $E_{(y)}$ be the species of sets, weighted by $w(U) = y^{|U|}$, and let $\wp_w$ be the species of subsets, weighted by $w(A) = y^{|A|}$, for $A \in \wp[U]$. In other words, the variable y acts as an element counter. Show that

a) $E_{(y)} = E(X_y)$ and $\wp_w = E \cdot E_{(y)}$,

b) $\wp_w(x) = \displaystyle\sum_{n \geq 0} (1 + y)^n \frac{x^n}{n!}$,

c) $\widetilde{\wp_w}(x) = \displaystyle\sum_{n \geq 0} \frac{1 - y^{n+1}}{1 - y} x^n$,

d) $Z_{\wp_w}(x_1, x_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} ((1+y)\,x_1)^{\sigma_1}((1+y^2)\,x_2)^{\sigma_2}\cdots,$

e) $|\mathrm{Fix}\,\wp_w[\sigma]| = (1+y)^{\sigma_1}(1+y^2)^{\sigma_2}\cdots,\quad$ for any permutation σ.

Exercises for Section 3.2

Exercise 3.14. Let $\Phi = \Phi(X, Y)$ be the two-sort species of **functions** $f : U \longrightarrow V$, where X is the sort of elements of U and Y that of elements of V. We have

$$\Phi[U, V] = \{f \mid f : U \longrightarrow V\}.$$

The transport of the corresponding structures is obtained by the usual composition of functions: $\Phi[\sigma, \theta](f) = \theta \circ f \circ \sigma^{-1}$. Moreover, we introduce the three subspecies Inj (**injections**), Sur (**surjections**), and Bij (**bijections**) of Φ by setting

$$
\begin{aligned}
\mathrm{Inj}[U, V] &= \{f \mid f : U \hookrightarrow V, \text{ i.e., } f \text{ injective}\}, \\
\mathrm{Sur}[U, V] &= \{f \mid f : U \twoheadrightarrow V, \text{ i.e., } f \text{ surjective}\}, \\
\mathrm{Bij}[U, V] &= \{f \mid f : U \xrightarrow{\sim} V, \text{ i.e., } f \text{ bijective}\}.
\end{aligned}
$$

a) Show that

i) $\Phi(X, Y) = E(E(X){\cdot}Y),\qquad$ ii) $\mathrm{Inj}(X, Y) = E((1+X){\cdot}Y),$

iii) $\mathrm{Sur}(X, Y) = E(E_+(X){\cdot}Y),\quad$ iv) $\mathrm{Bij}(X, Y) = E(X{\cdot}Y).$

b) Prove by combinatorial computations, as well as by geometric arguments, the following equalities (isomorphisms):

i) $\dfrac{\partial}{\partial X}\Phi = Y\dfrac{\partial}{\partial Y}\Phi,$

ii) $(1+X)\dfrac{\partial}{\partial X}\mathrm{Inj} = Y\dfrac{\partial}{\partial Y}\mathrm{Inj},$

iii) $\dfrac{\partial}{\partial X}\mathrm{Sur} = Y(1+\dfrac{\partial}{\partial Y})\mathrm{Sur},$

iv) $\dfrac{\partial}{\partial X}\mathrm{Bij} = Y\,\mathrm{Bij},\quad \dfrac{\partial}{\partial Y}\mathrm{Bij} = X\,\mathrm{Bij},$

v) $\mathrm{Bij}(X, Y) = \mathrm{Bij}(Y, X).$

c) Show that the series $\widetilde{\Phi} = \widetilde{\Phi}(x, y)$ and $Z_\Phi = Z_\Phi(x_1, x_2, x_3, \ldots; y_1, y_2, y_3, \ldots)$ are given by the formulas

i) $\widetilde{\Phi} = \dfrac{1}{(1-y)(1-xy)(1-x^2y)\cdots(1-x^ky)\cdots},$

ii) $Z_\Phi = \exp\left(\sum_{n\geq 1} \frac{y_n}{n} \exp \sum_{m\geq 1} \frac{x_{mn}}{m}\right).$

d) Compute explicitly the six other series: Z_{Inj}, Z_{Sur}, Z_{Bij}, $\widetilde{\mathrm{Inj}}$, $\widetilde{\mathrm{Sur}}$, and $\widetilde{\mathrm{Bij}}$.

e) More generally, for a species R on one sort, define the two-sort species $\Phi_R(X, Y) = E(R(X)\cdot Y)$, of **functions with R-enriched fibers**. Compute the series Z_{Φ_R} and $\widetilde{\Phi}_R$.

Exercise 3.15. Denote by $\mathrm{Oct}_{\mathrm{alt}} = \mathrm{Oct}_{\mathrm{alt}}(X, Y)$, the species of **alternating octopuses** on two sorts of elements X, Y, i.e., the octopuses on X and Y in which the adjacent elements are of different sort.

a) Verify that $\mathrm{Oct}_{\mathrm{alt}}(X, Y) = \mathcal{C}_{\mathrm{alt}}(X\cdot L(Y\cdot X)\cdot(1+Y), Y\cdot L(X\cdot Y)\cdot(1+X))$.

b) Deduce that

i) $\mathrm{Oct}_{\mathrm{alt}}(x, y) = \log \dfrac{(1-xy)^2}{1 - xy(3 + x + y)}$,

ii) $\widetilde{\mathrm{Oct}_{\mathrm{alt}}}(x, y) = \sum_{n\geq 1} \dfrac{\phi(n)}{n} \log \dfrac{(1 - x^n y^n)^2}{1 - x^n y^n(3 + x^n + y^n)}$,

iii) $Z_{\mathrm{Oct}_{\mathrm{alt}}} = \sum_{n\geq 1} \dfrac{\phi(n)}{n} \log \dfrac{(1 - x_n y_n)^2}{1 - x_n y_n(3 + x_n + y_n)}$.

Exercise 3.16. Consider the species $\mathcal{S}^{\mathrm{mix}}(X, Y)$ of permutations on two sorts X and Y, where each cycle has **at least** one element of each sort.

a) Show that $\mathcal{S}(X)\mathcal{S}(Y)\mathcal{S}^{\mathrm{mix}}(X, Y) = \mathcal{S}(X + Y)$.

b) Deduce the formulas:

i) $\mathcal{S}^{\mathrm{mix}}(x, y) = \dfrac{(1 - x)(1 - y)}{1 - x - y}$.

ii) $\widetilde{\mathcal{S}^{\mathrm{mix}}}(x, y) = \prod_{n\geq 1} \dfrac{(1 - x^n)(1 - y^n)}{1 - x^n - y^n}$,

iii) $Z_{\mathcal{S}^{\mathrm{mix}}} = \prod_{n\geq 1} \dfrac{(1 - x_n)(1 - y_n)}{1 - x_n - y_n}$.

Exercise 3.17. Consider the species $\Gamma = \Gamma(X, Y)$ of graphs on vertices of sort X, and of edges of sort Y, introduced in Example 3.32, as well as the species of multigraphs $\Gamma_{\mathrm{mult}} = \Gamma_{\mathrm{mult}}(X, Y)$, where multiple edges are permitted.

a) Verify the following equalities

i) $\Gamma(x, y) = \mathrm{T}_{Y;y}\Gamma(x) = \sum_{n \geq 0}(1 + y)^{\binom{n}{2}} \dfrac{x^n}{n!}$,

ii) $\Gamma_{\mathrm{mult}}(x, y) = \sum_{n \geq 0}(e^y)^{\binom{n}{2}} \dfrac{x^n}{n!}$,

iii) $\mathrm{T}_{Y;y}\Gamma_{\mathrm{mult}}(x) = \sum_{n \geq 0}\left(\dfrac{1}{1-y}\right)^{\binom{n}{2}} \dfrac{x^n}{n!}$.

b) Show that

  i) $\Gamma(X, Y) = \mathrm{Inj} \,\square\,(\varepsilon(Y), \wp^{[2]}(X))$,

  ii) $\Gamma_{\mathrm{mult}}(X, Y) = \Phi \,\square\,(\varepsilon(Y), \wp^{[2]}(X))$,

  where ε denotes the species of elements (see Section 1.1), and $\wp^{[2]}$, that of subsets with two elements (see Section 2.4).

c) More generally, for a unisort species R, define the two-sort species of multigraphs with an R-enrichment on the edges having the same endpoints, by $\Gamma_R(X, Y) = \Phi_R \,\square\,(\varepsilon(Y), \wp^{[2]}(X))$ (see Exercise 3.14). Verify that

$$\Gamma_R(x, y) = \sum_{n \geq 0} R(y)^{\binom{n}{2}} \frac{x^n}{n!}.$$

Exercise 3.18. Consider the two-sort species $\mathcal{B} = \mathcal{B}(X, T)$, of rooted trees on internal vertices of sort X and leaves of sort Y (see Figure 3.5). Let $\mathcal{A}_w = \mathcal{A}_w(X)$ be the weighted species of rooted trees on vertices of sort X, the weight of each rooted tree α being given by $w(\alpha) = t^f$, where f is the number of leaves of α. Verify the combinatorial equality $\mathcal{A}_w(X) = \mathcal{B}(X, X_t)$, where X_t is the species of singletons of weight t. Do not forget transports of structures.

Exercise 3.19. FUNCTORIAL POINT OF VIEW. Let k be an integer ≥ 1. Denote by $\mathbb{B}$, the category of finite sets and bijections, and by $\mathbb{B}^k = \mathbb{B} \times \cdots \times \mathbb{B}$, the category, product of k copies of $\mathbb{B}$, of finite multisets (of k sorts) and of multibijections. Consider also the category, denoted by $\Phi(\mathbb{B}, [k])$, whose objects are the pairs (U, χ), where U is a finite set and $\chi : U \longrightarrow [k]$ is a function, and whose morphisms $\sigma : (U, \chi) \longrightarrow (V, \psi)$ are the bijections $\sigma : U \longrightarrow V$ such that $\psi \circ \sigma = \chi$. Let $\mathcal{R}$ be a ring of formal power series on a ring $\mathbb{K} \subseteq \mathbb{C}$. Denote by $\mathbb{E}_\mathcal{R}$, the category of summable $\mathcal{R}$-weighted sets and $\mathcal{R}$-weighted morphisms of sets (see Definitions 3.2 and 3.3).

a) Show that the categories $\mathbb{B}^k$ and $\Phi(\mathbb{B}, [k])$ are **equivalent** by explicitly describing functors $R : \mathbb{B}^k \longrightarrow \Phi(\mathbb{B}, [k])$ and $S : \Phi(\mathbb{B}, [k]) \longrightarrow \mathbb{B}^k$ as well as mutually inverse natural transformations $\alpha : S \circ R \longrightarrow \mathrm{Id}_{\mathbb{B}^k}$ and $\beta : R \circ S \longrightarrow \mathrm{Id}_{\Phi(\mathbb{B}, [k])}$.

b) Show that a weighted k-sort species $F = F_v$ can be considered as a functor $F : \mathbb{B}^k \longrightarrow \mathbb{E}_\mathcal{R}$ or equivalently, as a functor $F : \Phi(\mathbb{B}, [k]) \longrightarrow \mathbb{E}_\mathcal{R}$.

Exercise 3.20.   a) Complete the definitions of the combinatorial operations $(+, \cdot, \circ, \times, \square)$ on species in the weighted multisort context, not forgetting transports of structures in each case.

b) State and prove the properties of these combinatorial operations, in analogy with formulas of Esercises 2.1, 2.4, 2.14, 2.44, and 2.53.

c) Describe the behavior of generating and index series with respect to these operations. In particular, prove formulas (3.13) and (3.14).

d) Prove formulas (3.15) for the series associated to the species of types according to sort.

e) Extend these results to multisort species.

Exercise 3.21.   a) Let $F = F(X_1, X_2, \ldots, X_k)$ and $G = G(X_1, X_2, \ldots, X_k)$, be species on many sorts X_1, X_2, …, X_k. Prove for $i = 1, \ldots, k$ the equalities

i) $\dfrac{\partial}{\partial X_i}(F + G) = \dfrac{\partial}{\partial X_i}F + \dfrac{\partial}{\partial X_i}G,$

ii) $\dfrac{\partial}{\partial X_i}(F \cdot G) = \left(\dfrac{\partial}{\partial X_i}F\right) \cdot G + F \cdot \left(\dfrac{\partial}{\partial X_i}G\right),$

iii) $\dfrac{\partial^2}{\partial X_i \partial X_j}F = \dfrac{\partial^2}{\partial X_j \partial X_i}F.$

b) Consider the species $F = F(Y_1, Y_2, \ldots, Y_m)$ and $G_j = G_j(X_1, X_2, \ldots, X_k)$, $j = 1, \ldots, m$. Prove the chain rule

$$\frac{\partial}{\partial X_i}F(G_1, G_2, \ldots, G_m) = \sum_{j=1}^{m} \frac{\partial F}{\partial Y_j}(G_1, G_2, \ldots, G_m) \cdot \frac{\partial}{\partial X_i}G_j.$$

Exercise 3.22. Consider a species on many sorts $F = F(X_1, X_2, \ldots, X_k)$. Interpret combinatorially the following species:

$$\text{a) } \sum_{i=1}^{k} X_i \frac{\partial}{\partial X_i}F, \qquad \text{b) } X_2 \frac{\partial}{\partial X_1}F, \qquad \text{and} \qquad \text{c) } \sum_{i \neq j} X_i \frac{\partial}{\partial X_j}F.$$

Exercise 3.23. Show that the species $\mathcal{G}_{\text{chro}} = \mathcal{G}_{\text{chro}}(X, Y, Z)$ of tri-chromatic graphs (see Example 3.17) is not of the form $F(X + Y + Z)$, where F is a unisort species.

Exercise 3.24. Write explicitly the three identities between generating and cycle index series which correspond to the combinatorial equality $\mathcal{C}(X + Y) = \mathcal{C}(X) + \mathcal{C}(Y \cdot L(X))$.

Exercise 3.25.   a) Show that by taking the types according to Y of the species $\mathrm{Sur}(X, Y)$ of surjections, one obtains the species $\mathrm{Par}_v(X)$ of partitions weighted by $v(\pi) = y^{b(\pi)}$, where $b(\pi)$ is the number of blocks of π, i.e., $\mathrm{Par}_v(X) = \mathrm{T}_{Y;y}\mathrm{Sur}(X, Y)$.

b) Deduce that the Stirling numbers of the second kind $S(n, k)$ are given by $S(n, k) = \dfrac{|\mathrm{Sur}[n, k]|}{k!}.$

c) Establish the combinatorial identity $\Phi(X, Y) = \mathrm{Sur}(X, Y) \cdot E(Y)$ and deduce Touchard's formula,

$$\sum_{n,k \geq 0} k^n \frac{x^n}{n!} \frac{y^k}{k!} = \sum_{n,k \geq 0} S(n,k) \frac{x^n}{n!} y^k \cdot \sum_{j \geq 0} \frac{y^j}{j!}.$$

d) Deduce Dobinski's formula for Bell numbers,

$$B_n = \frac{1}{e} \sum_{k \geq 0} \frac{k^n}{k!}, \quad n \geq 0.$$

Exercise 3.26. Consider the species $\mathcal{S}(X, Y) = \mathcal{S}(X + Y)$ of permutations of elements of sort X and elements of sort Y.

a) Describe, using a geometric figure, a typical $\mathrm{T}_Y \mathcal{S}$-structure on a finite set U.

b) Compute the three series $(\mathrm{T}_{Y;y} S)(x)$, $(\widetilde{\mathrm{T}_{Y;y} S})(x)$ and $Z_{\mathrm{T}_{Y;y} S}(x_1, x_2, x_3, \ldots)$.

Exercise 3.27.   a) Show that the species $\mathcal{B} = \mathcal{B}(X, T)$ of Exercise 5 is not polynomial in X nor in T.

b) Let k be an integer ≥ 1, and consider the subspecies F of $\mathcal{B}$ formed of rooted trees in which each internal vertex is of degree $\leq k$. Show that $F = F(X, T)$ is polynomial in T but not in X.

c) Let k be an integer ≥ 1, and consider the subspecies G of $\mathcal{B}$ formed of rooted trees in which each internal vertex is of degree between 2 and k. Show that $F = F(X, T)$ is polynomial both in T and in X.

Exercise 3.28. Let $F_v = F_v(X_1, \ldots, X_k)$ be an $\mathcal{R}$-weighted multisort species. Let i be an integer such that $1 \leq i \leq k$.

a) Define the type $\mathrm{T}_{X_i} s$ of an F_v-structure s according to the sort X_i, as well as its weight $w(\mathrm{T}_{X_i} s) \in \mathcal{R}[\![x_i]\!]$.

b) Define the $\mathcal{R}[\![x_i]\!]$-weighted species $\mathrm{T}_{X_i;x_i} F_v$ of types of F_v-structures according to the sort X_i.

c) Give finiteness conditions ensuring existence of the species $\mathrm{T}_{X_i} F_v = \mathrm{T}_{X_i;x_i} F_v \big|_{x_i = 1}$.

Exercise 3.29. GENERALIZED POINTING. Consider two species of structures $F = F(X)$ and $G = G(X)$. Define the F-**pointing** of G as being the species, denoted by $F \langle\!\langle X \frac{d}{dX} \rangle\!\rangle G(X)$, given by

$$F \langle\!\langle X \tfrac{d}{dX} \rangle\!\rangle G(X) = (F(X) \cdot E(X)) \times G(X).$$

a) Give a combinatorial (geometrical) description of this definition.

b) Given a polynomial $p(u,v,w,\ldots) = \sum_{i,j,k,\ldots} p_{i,j,k,\ldots}\, u^i v^j w^k \cdots$, let us agree to write

$$p\langle\!\langle u,v,w,\ldots\rangle\!\rangle = \sum_{i,j,k,\ldots} p_{i,j,k,\ldots}\, u_{<i>} v_{<j>} w_{<k>} \cdots ,$$

where $x_{<n>} = x\,(x-1)\,(x-2)\cdots(x-n+1)$. Suppose that F is polynomial in X (see Exercise 19) and set $H(X) = F\langle\!\langle X\frac{d}{dX}\rangle\!\rangle G(X)$. Establish the equalities

i) $H(x) = F\langle\!\langle x\frac{d}{dx}\rangle\!\rangle G(x)$,

ii) $\widetilde{H}(x) = Z_F\langle\!\langle x_1\frac{\partial}{\partial x_1}, 2\,x_2\frac{\partial}{\partial x_2}, 3\,x_3\frac{\partial}{\partial x_3},\ldots\rangle\!\rangle Z_G\Big|_{x_i:=x^i,\ i\geq1}$,

iii) $Z_H(x_1,x_2,x_3,\ldots) = Z_F\langle\!\langle x_1\frac{\partial}{\partial x_1}, 2\,x_2\frac{\partial}{\partial x_2}, 3\,x_3\frac{\partial}{\partial x_3},\ldots\rangle\!\rangle Z_G(x_1,x_2,x_3,\ldots)$.

Exercise 3.30.   a) Show that the species $\Gamma = \Gamma(X,Y)$, $\Gamma_{\mathrm{mult}} = \Gamma_{\mathrm{mult}}(X,Y)$ and $\Gamma_R = \Gamma_R(X,Y)$, of Exercise 4, satisfy the combinatorial equations (see Knuth [155])

i) $(1+Y)\dfrac{\partial}{\partial Y}\Gamma(X,Y) = E_2\langle\!\langle X\frac{\partial}{\partial X}\rangle\!\rangle\Gamma(X,Y)$,

ii) $\dfrac{\partial}{\partial Y}\Gamma_{\mathrm{mult}}(X,Y) = E_2\langle\!\langle X\frac{\partial}{\partial X}\rangle\!\rangle\Gamma_{\mathrm{mult}}(X,Y)$,

iii) $R(Y)\dfrac{\partial}{\partial Y}\Gamma_R(X,Y) = R'(Y)E_2\langle\!\langle X\frac{\partial}{\partial X}\rangle\!\rangle\Gamma_R(X,Y)$.

b) Write the implied relations for the generating and index series.

Exercise 3.31. GENERALIZED TAYLOR FORMULA. Given two species of structures $F = F(X)$ and $G = G(X)$, we define the two-sort (X and T) species $F(T\frac{\partial}{\partial X})G(X)$ by setting

$$F(T\tfrac{\partial}{\partial X})G(X) = (E(X)\!\cdot\! F(T)) \times G(X+T).$$

a) Give a geometrical description of this combinatorially definition for structures of species $F(T\partial/\partial X)G(X)$ on a pair of finite sets (U,V).

b) Set $H(X,T) = F(T\frac{\partial}{\partial X})G(X)$. Establish combinatorially the following relations:

i) $H(x,t) = F(t\frac{\partial}{\partial x})G(x)$,

ii) $\widetilde{H}(x,t) = (Z_F(t_1\frac{\partial}{\partial x_1}, 2\,t_2\frac{\partial}{\partial x_2}, 3\,t_3\frac{\partial}{\partial x_3},\ldots)Z_G)\Big|_{x_i:=x^i,\ t_i:=t^i,\ i=1,2,\ldots}$,

iii) $Z_H(x_1,x_2,\ldots;t_1,t_2,\ldots) = Z_F(t_1\frac{\partial}{\partial x_1}, 2\,t_2\frac{\partial}{\partial x_2},\ldots)Z_G(x_1,x_2,\ldots)$.

The F-**Taylor expansion** of $G(X)$ in T is defined by the formula

$$(E(X)\!\cdot\! F(T)) \times G(X+T) = \sum_{k\geq0} F_k(T\tfrac{\partial}{\partial X})G(X)$$

where $F = F_0 + F_1 + F_2 + \ldots$ is the canonical decomposition of F. By letting $X := 0$ above we obtain, by definition, the F-**Maclaurin expansion** of $G(X)$ in T:

$$F(T) \times G(T) = \sum_{k\geq0} F_k(T\tfrac{\partial}{\partial X})G(X)\big|_{X:=0}.$$

c) Show that the Taylor expansion and Maclaurin expansion above are indeed summable and prove the combinatorial identities. In particular, when $F = E$, show that these series take the form:

$$G(X + T) = G(X) + T\,G'(X) + E_2(T\tfrac{\partial}{\partial X})G(X) + \ldots + E_k(T\tfrac{\partial}{\partial X})G(X) + \ldots.$$

d) Verify that the following "classical" formulas are satisfied:

i) $G(x + t) = e^{t\frac{\partial}{\partial x}}G(x) = G(x) + t\,G'(x) + \dfrac{t^2}{2!}\,G''(x) + \ldots + \dfrac{t^k}{k!}\,G^{(k)}(x) + \ldots,$

ii) $G(t) = e^{t\frac{\partial}{\partial x}}G(x)\Big|_{x:=0} = G(0) + t\,G'(0) + \dfrac{t^2}{2!}\,G''(0) + \ldots + \dfrac{t^k}{k!}\,G^{(k)}(0) + \ldots,$

iii) $Z_{G(X+T)}(x_1, x_2, \ldots; t_1, t_2, \ldots) = Z_G(x_1+t_1, x_2+t_2, \ldots) = e^{t_1\frac{\partial}{\partial x_1}+t_2\frac{\partial}{\partial x_2}+\cdots}Z_G(x_1, x_2, \ldots),$

iv) $Z_G(t_1, t_2, \ldots) = e^{t_1\frac{\partial}{\partial x_1}+t_2\frac{\partial}{\partial x_2}+\cdots}Z_G(x_1, x_2, \ldots)\Big|_{x_i:=0,\ i=1,2,\ldots}.$

Exercise 3.32. GENERALIZED DIFFERENTIATION. Let $F = F(X)$ be a polynomial species in X (i.e., there exists $N \geq 0$ such that $|U| > N \implies F[U] = \emptyset$). Given a species $G = G(X)$, define the species $F(\tfrac{d}{dX})G(X)$ by setting

$$F(\tfrac{d}{dX})G(X) = \mathrm{T}_Y\left[(E(X)\cdot F(Y)) \times G(X + Y)\right],$$

where Y is an auxiliary sort different from X.

a) Show that the species $(E(X)\cdot F(Y)) \times G(X + Y)$ is polynomial in Y and interpret combinatorially the definition above by describing geometrically an $F(\tfrac{d}{dX})G(X)$-structure on a finite set U.

b) Set $H(X) = F(\tfrac{d}{dX})G(X)$. Establish the relations

i) $H(x) = F(\tfrac{d}{dx})G(x)$, if $F = X^n$, but $H(x) \neq F(\tfrac{d}{dx})G(x)$ in the general case,

ii) $\widetilde{H}(x) = \{Z_F(\tfrac{\partial}{\partial x_1}, 2\tfrac{\partial}{\partial x_2}, 3\tfrac{\partial}{\partial x_3}, \ldots)Z_G\}\big|_{x_i:=x^i,\ i\geq 1}$,

iii) $Z_H(x_1, x_2, x_3, \ldots) = Z_F(\tfrac{\partial}{\partial x_1}, 2\tfrac{\partial}{\partial x_2}, 3\tfrac{\partial}{\partial x_3}, \ldots)Z_G(x_1, x_2, x_3, \ldots).$

Bibliography

[1] A.V. Aho, J.E. Hopcroft, and J.D. Ullman, *Data Structures and Algorithms*, Addison–Wesley, Reading, MA, **1983**.

[2] D. André, *Sur les permutations alternées*, **Journal de mathématiques pures et appliquées**, 5, **1879**, 31–46.

[3] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer–Verlag, Berlin, Heidelberg, and New York, **1976**.

[4] P. Auger, *Étude exacte et asymptotique de divers paramètres de structures arborescentes*, M.Sc. Thesis, Université du Québec à Montréal, **1993**.

[5] R. Baeza-Yates and G.H. Gonnet, *Handbook of Algorithms and Data Structures: in Pascal and C*, 2nd, Addison–Wesley, Reading, MA, **1991**.

[6] M. Bajraktarević, *Sur une équation fonctionnelle*, **Glasnik Matematičko-Fizički I Astronomski**, 12, **1957**, 201–205.

[7] F. Bédard and A. Goupil, *The Lattice of Conjugacy Classes of the Symmetric Group*, **Canadian Bulletin of Mathematics**, 35, **1992**, 152–160.

[8] E.A. Bender, *Central and Local Limit Theorems Applied to Asymptotic Enumeration*, **Journal of Combinatorial Theory, Series A**, 15, **1973**, 91–111.

[9] E.A. Bender, *Asymptotic Methods in Enumeration*, **SIAM Review**, 16, **1974**, 485–515.

[10] E.A. Bender and E.R. Canfield, *The Asymptotic Number of Labeled Connected Graphs With a Given Number of Vertices and Edges*, **Random Structures and Algorithms**, 1, **1990**, 127–169.

[11] E.A. Bender and J.R. Goldman, *Enumerative Uses of Generating Functions*, **Indiana University Mathematics Journal**, 20, **1971**, 753–765.

[12] E.A. Bender and S.G. Williamson, *Foundations of Applied Combinatorics*, Addison–Wesley, Reading, MA, **1991**.

[13] C. Berge, *Principles of combinatorics*, Academic, New York, **1970**.

[14] F. Bergeron, *Une systématique de la combinatoire énumérative*, Ph.D. Dissertation, Université de Montréal, Canada, **1985**.

[15] F. BERGERON, *Une approche combinatoire de la méthode de Weisner*, in *Polynômes orthogonaux et applications*, editors: C. Brezinski, A. Draux, A.P. Magnus, P. Maroni, and A. Ronveaux, Lecture Notes in Mathematics 1171, Springer–Verlag, Berlin, Heidelberg, and New York, **1985**, 111–119.

[16] F. BERGERON, *Combinatorial Representations of Lie Groups and Algebras*, in [196], **1986**, 34–47.

[17] F. BERGERON, *Une combinatoire du pléthysme*, **Journal of Combinatorial Theory, Series A**, 46, **1987**, 291–305.

[18] F. BERGERON, *A Combinatorial Outlook on Symmetric Functions*, **Journal of Combinatorial Theory, Series A**, 50, **1989**, 226–234.

[19] F. BERGERON, *Algorithms for the Sequential Generation of Combinatorial Structures*, **Discrete Applied Mathematics**, 24, **1989**, 29–35.

[20] F. BERGERON, *Combinatoire des polynômes orthogonaux classiques: une approche unifiée*, **European Journal of Combinatorics**, 11, **1990**, 393–401.

[21] F. BERGERON, P. FLAJOLET, AND B. SALVY, *Varieties of Increasing Trees*, in *Proceedings of the 17th Colloquium on Trees in Algebra and Programming*, Ed. J.-C. Raoult, Lecture Notes in Computer Sciences 581, Springer–Verlag, Berlin, Heidelberg, and New York, **1991**, 24–48.

[22] F. BERGERON, G. LABELLE, AND P. LEROUX, *Combinatorial Species and Tree-Like Structures*, **Cambridge University Press**, **1998**.

[23] F. BERGERON, G. LABELLE, AND P. LEROUX, *Functional Equations for Data Structures*, in *Proceedings of STACS-88*, editors: R. Cori and M. Wirsing, Lecture Notes in Computer Sciences 294, Springer–Verlag, Berlin, Heidelberg, and New York, **1988**, 73–80.

[24] F. BERGERON, G. LABELLE, AND P. LEROUX, *Computation of the Expected Number of Leaves in a Tree Having a Given Automorphism*, **Discrete Applied Mathematics**, 34, **1991**, 49–66.

[25] F. BERGERON AND S. PLOUFFE, *Computing the Generating Function of a Series Given its First Few Terms*, **Experimental Mathematics**, 1, **1992**, 307–312.

[26] F. BERGERON AND C. REUTENAUER, *Interprétation combinatoire des puissances d'un opérateur différentiel linéaire*, **Annales des Sciences Mathématiques du Québec**, 11, **1987**, 269–278.

[27] F. BERGERON AND C. REUTENAUER, *Combinatorial Resolution of Systems of Differential Equations III: A Special Class of Differentiably Algebraic Series*, **European Journal of Combinatorics**, 11, **1990**, 501–512.

[28] F. BERGERON AND U. SATTLER, *Constructibly Differentially Algebraic Series in Several Variables*, **Theoretical Computer Science**, 144, **1995**, 59–65.

[29] F. BERGERON AND Y.N. YEH, *The Factoriality of the Ring of s-Species*, **Journal of Combinatorial Theory, Series A**, 55, **1990**, 194–203.

[30] J. BERSTEL AND C. REUTENAUER, *Zeta Functions of Formal Languages*, **Transactions of the American Mathematical Society**, 321, **1990**, 533–546.

[31] J. Bétréma and A. Zvonkin, *Planes Trees and Shabat Polynomials*, in **Discrete Mathematics**, 153, **1996**, 47–58.

[32] F. Bonetti, G.C. Rota, D. Senato, and A.M. Venezia, *Symmetric Functions and Symmetric Species*, *Combinatorics'84* (Bari, 1984), North–Holland Mathematical Studies 123, Amsterdam, **1986**, 107–113.

[33] F. Bonetti, G.C. Rota, D. Senato, and A.M. Venezia, *On the Foundation of Combinatorial Theory X, A Categorical Setting for Symmetric Functions*, **Studies in Applied Mathematics**, 86, **1992**, 1–29.

[34] K.S. Booth and G.S. Lueker, *Testing for the Consecutive Ones Property, Interval Graphs, and Graph Planarity using PQ-Tree Algorithms*, **Journal of Computer and System Sciences**, 13, **1976**, 335–379.

[35] P. Bouchard, Y. Chiricota, and G. Labelle, *Arbres, arborescences et racines carrées symétriques*, **Discrete Mathematics**, 139, **1995**, 49–56.

[36] P. Bouchard and M. Ouellette, *Décomposition arborescente de Mario Ouellette pour les espèces de structures*, *Actes du séminaire lotharingien de combinatoire*, Publications de l'Institut de recherche mathématiques, Strasbourg, France, **1990**, 5–13.

[37] N. Bourbaki, *Anneaux topologiques*, Herman, Paris, **1971**.

[38] G. Brassard and P. Bratley, *Algorithmique, conception et analyse*, Masson/Les Presses de Université de Montréal, Canada, **1987**.

[39] R.P. Brent and H.T. Kung, *Fast Algorithms for Manipulating Formal Power Series*, **Journal of the ACM**, 25, **1978**, 581–595.

[40] T.J. l'a Bromwich, *Introduction to the Theory of Infinite Series*, Macmillan, New York, **1965**.

[41] W.H. Burge, *An Analysis of a Tree Sorting Method and Some Properties of a Set of Trees*, in *Proceedings of the First USA-Japan Computer Conference*, **1972**, 372–379.

[42] W. Burnside, *Theory of Groups of Finite Order*, Dover, New York, **1955**.

[43] E.R. Canfield, *Remarks on an Asymptotic Method in Combinatorics*, **Journal of Combinatorial Theory, Series A**, 37, **1984**, 348–352.

[44] G. Cartier, *Darwin, Manuel d'utilisateur*, **Publications du Laboratoire de combinatoire et d'informatique mathématique**, Université du Québec à Montréal, **1990**.

[45] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et de réarrangements*, Lecture Notes in Mathematics 85, Springer–Verlag, Berlin, Heidelberg, and New York, **1969**.

[46] A. Cayley, *A Theorem on Trees*, **Quarterly Journal of Mathematics**, Oxford Series 23, **1889**, 376-378; Collected papers, Cambridge 13, **1897**, 26–28.

[47] B.W. Char, K.O. Geddes, G.H. Gonnet, M.B. Monagan, and S.M. Watt, *Maple Reference Manual, 5th edition*, Watcom, Waterloo, Ontario, **1988**.

[48] W.Y.C. Chen, *A General Bijective Algorithm for Trees*, **Proceedings of the National Academy of Sciences USA** , 87, **1990**, 9635–9639.

[49] W.Y.C. Chen, *The Theory of Compositionals*, **Discrete Mathematics**, 22, **1993**, 59–87.

[50] W.Y.C. Chen, *Compositional Calculus*, **Journal of Combinatorial Theory, Series A**, 64, **1993**, 149–188.

[51] W.Y.C. Chen, *A Bijection for Enriched Trees*, **European Journal of Combinatorics**, 15, **1994**, 337–343.

[52] N. Chiba, T. Nishizeki, S. Abe, and T. Ozawa, *A Linear Algorithm for Embedding Planar Graphs Using PQ-Trees*, **Journal of Computer and System Sciences**, 30, **1985**, 54–76.

[53] I. Chiricota, *Structures combinatoires et calcul symbolique*, Ph.D. Dissertation, Université du Québec à Montréal, **Publications du Laboratoire de combinatoire et d'informatique mathématique** Vol. 12, **1993**.

[54] Y. Chiricota, *Classification des espèces moléculaires de degré 6 et 7*, **Annales des Sciences Mathématiques du Québec**, 17, **1993**, 11–37.

[55] Y. Chiricota and G. Labelle, *Familles de solutions combinatoires de l'équation différentielle $Y'=1+Y^2$ et des équations différentielles autonomes*, **Discrete Mathematics**, 115, **1993**, 77–93.

[56] Y. Chiricota and V. Strehl, *Inversion of Polynomial Species and Jacobian Conjecture*, **Bayreuther Mathematische Schriften**, 44, **1993**, 53–61.

[57] L. Chottin, *Énumération d'arbres et formules d'inversion de séries formelles*, **Journal of Combinatorial Theory, Series B**, 31, **1981**, 23–45.

[58] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, The Netherlands, **1974**.

[59] I. Constantineau, *Calcul combinatoire de séries indicatrices de cycles*, Ph.D. Dissertation, Université du Québec à Montréal, **Publications du Laboratoire de combinatoire et d'informatique mathématique** Vol. 5, **1991**.

[60] I. Constantineau, *Auto-similarité dans la combinatoire des polynômes orthogonaux*, **Theoretical Computer Science**, 117, **1993**, 153–167.

[61] I. Constantineau, *Le nombre d'arbres m-Husimi invariants sous une permutation des sommets*, **Discrete Mathematics**, 139, **1995**, 89–103.

[62] I. Constantineau and G. Labelle, *Une généralisation automorphe des nombres de Stirling*, **Discrete Mathematics**, 157, **1996**, 53-64.

[63] I. Constantineau and J. Labelle, *Le nombres d'endofonctions et d'arborescences laissées fixes par l'action d'une permutation*, **Annales des Sciences Mathématiques du Québec**, 13, **1989**, 33–38.

[64] I. Constantineau and J. Labelle, *On combinatorial structures kept fixed by the action of a given permutation*, **Studies in Applied Mathematics**, 84, **1991**, 105–118.

[65] I. Constantineau and J. Labelle, *On the Construction of Permutations of a Given Type Kept Fixed by Conjugation*, **Journal of Combinatorial Theory, Series A**, 62, **1993**, 199–209.

[66] M. Content, F. Lemay, and P. Leroux, *Catégories de Möbius et fonctorialité, un cadre pour l'inversion de Möbius*, **Journal of Combinatorial Theory, Series A**, 28, **1980**, 169–190.

[67] R. Cori and B. Vauquelin, *Planar Maps are Well Labelled Trees*, **Canadian Journal of Mathematics**, 33, **1981**, 1023–1042.

[68] N.G. de Bruijn, *Enumerative Combinatorial Problems Concerning Structures*, **Nieuw Archief voor Wiskunde** (3), XI, **1963**, 142–161.

[69] N.G. de Bruijn, *Asymptotic Methods in Analysis*, Dover, New York, **1981**.

[70] H. Décoste, *Séries indicatrices d'espèces pondérées et q-analogues*, Ph.D. Dissertation, Université de Montréal, Canada, **1989**. **Publications du Laboratoire de combinatoire et d'informatique mathématique** Vol. 2, Université du Québec à Montréal, **1990**.

[71] H. Décoste, *Séries indicatrices et q-series*, **Theoretical Computer Science**, 117, **1993**, 169–186.

[72] H. Décoste and G. Labelle, *Le q-dénombrement générique d'une espèce: existence et méthode de calcul*, **Discrete Mathematics**, 153, **1996**, 59–67.

[73] H. Décoste, G. Labelle, and P. Leroux, *Une approche combinatoire pour l'itération de Newton-Raphson*, **Advances in Applied Mathematics**, 3, **1982**, 407–416.

[74] H. Décoste, G. Labelle, and P. Leroux, *The Functorial Composition of Species, a Forgotten Operation*, **Discrete Mathematics**, 99, **1992**, 31–48.

[75] M. Delest and G.X. Viennot, *Algebraic Languages and Polyomino Enumeration*, **Theoretical Computer Science**, 34, **1984**, 169–206.

[76] L. Devroye, *Applications of the Theory of Records in the Study of Random Trees*, **Acta Informatica**, 26, **1988**, 123–130.

[77] L. Devroye and L. Laforest, *An Analysis of Random d-Dimensional Quad Trees*, **SIAM Journal of Computing**, 19, **1990**, 821–832.

[78] V. Domocos and W.R. Schmitt, *An Application of Linear Species*, **Discrete Mathematics**, 132, **1994**, 377–381.

[79] W.F. Doran IV, *The Lattice of Periods of a Group Action*, **Advances in Mathematics**, 110, **1995**, 88–108.

[80] P. Doubilet, G.-C. Rota, and R.P. Stanley, *The Idea of Generating Function*, in *Finite Operator Calculus*, Ed. G.-C. Rota, Academic, New York, **1975**, 83–134.

[81] S. Dubuc, *Une équation fonctionnelle pour diverses constructions géométriques*, **Annales des Sciences Mathématiques du Québec**, 9, **1985**, 407–416.

[82] S. Dubuc and A. Elqortobi, *Enveloppe convexe d'une courbe de Mandelbrot*, **Annales des Sciences Mathématiques du Québec**, 11, **1987**, 45–64.

[83] D. Dumont, *Une approche combinatoire des fonctions elliptiques de Jacobi*, **Advances in Mathematics**, 41, **1981**, 1–39.

[84] A. Dür, *Mobius Functions, Incidence Algebras and Power Series Representations*, Lecture Notes in Mathematics 1202, Springer–Verlag, Berlin, Heidelberg, and New York, **1986**.

[85] P.H. Edelman, R. Simion, and D. White, *Partition Statistics on Permutations*, **Discrete Mathematics**, 99, **1992**, 63–68.

[86] R. Ehrenborg and M. Méndez, *A Bijective Proof of Infinite Variated Good's Inversion*, **Advances in Mathematics**, 103, **1994**, 221–259.

[87] C. Ehresmann, *Catégories et structures*, Dunod, Paris, **1965**.

[88] S. Eilenberg, *Automata, Languages and Machines*, Academic, New York, Vol. A, **1974**, Vol. B, **1976**.

[89] P. Fatou, *Sur les équations fonctionnelles*, **Bulletin de la société mathématique de France**, 48, **1920**, 208–314.

[90] G. Fayole, P. Flajolet, and M. Hofri, *On a Functional Equation Arising in the Analysis of a Protocol for Multi-Access Broadcast Channel*, **Advances in Applied Probability**, 18, **1986**, 441–472.

[91] P. Flajolet, *Elements of a General Theory of Combinatorial Structures*, in *Fundamentals of Computation Theory*, Ed. L. Budach, Lecture Notes in Computer Sciences 199, Springer–Verlag, Berlin, Heidelberg, and New York, **1985**, 112–127.

[92] P. Flajolet, *Random Tree Models in the Analysis of Algorithms*, in *Performance'87*, editors: P.-J. Courtois and G. Latouche, Elsevier Science Publishers, New York, **1988**, 171–187.

[93] P. Flajolet and J. Françon, *Elliptic Functions, Continued Fractions and Doubled Permutations*, **European Journal of Combinatorics**, 10, **1989**, 235–241.

[94] P. Flajolet, G. Gonnet, C. Puech, and J.M. Robson, *Analytic Variations on Quad Trees*, **Algorithmica**, 10, **1993**, 473–500.

[95] P. Flajolet, G. Labelle, and L. Laforest, *Hypergeometrics and the Cost Structure of Quadtrees*, **Random Structures and Algorithms**, 7, **1995**, 117–144.

[96] P. Flajolet and A.M. Odlyzko, *Limit Distributions for Coefficients of Iterates of Polynomials with Applications to Combinatorial Enumerations*, **Proceedings of the Cambridge Philosophical Society**, 96, **1984**, 237–253.

[97] P. Flajolet and A.M. Odlyzko, *Singularity Analysis of Generating Functions*, **SIAM Journal on Discrete Mathematics**, 3, **1990**, 216–240.

[98] P. Flajolet and H. Prodinger, *Level Number Sequences for Trees*, **Discrete Mathematics**, 65, **1987**, 149–156.

[99] P. FLAJOLET, B. SALVY, AND P. ZIMMERMANN, *Automatic Average-case Analysis of Algorithms*, **Theoretical Computer Science**, 79, **1991**, 37–109.

[100] P. FLAJOLET AND M. SORIA, *Gaussian Limiting Distributions for the Number of Components in Combinatorial Structures*, **Journal of Combinatorial Theory, Series A**, 53, **1990**, 165–182.

[101] P. FLAJOLET AND J.M. STEYAERT, *Patterns and Pattern Matching in Trees, an Analysis*, **Information Control**, 58, **1988**, 19–58.

[102] P. FLAJOLET AND J.S. VITTER, *Analysis of Algorithms and Data Structures*, in *Handbook of Theoretical Computer Science*, Ed. J. van Leeuwen, Vol. A: *Algorithms and Complexity*, North Holland, Amsterdam, **1990**, 431–524.

[103] M. FLIESS, *Fonctionnelles causales non linéaires et indéterminées non commutatives*, **Bulletin de la société mathématique de France**, 109, **1981**, 3–40.

[104] M. FLIESS AND F. LAMNABHI–LAGARRIGUE, *Application of a New Functional Expansion to the Cubic Anharmonic Oscillator*, **Journal of Mathematical Physics**, 23, **1982**, 495–502.

[105] D. FLORES-DE-CHELA AND M.A. MENDEZ, *Equivariant Representations of the Symmetric Group*, **Advances in Mathematics**, 97, **1993**, 191–230.

[106] D. FOATA, *La série génératrice exponentielle dans les problèmes d'énumération*, Presses de l'Université de Montréal, Montréal, Québec, **1974**.

[107] D. FOATA, *A Combinatorial Proof of the Mehler Formula*, **Journal of Combinatorial Theory, Series A**, 24, **1978**, 367–376.

[108] D. FOATA, *A Combinatorial Proof of Jacobi's Identity*, **Annals of Discrete Mathematics**, 6, **1980**, 125–135.

[109] D. FOATA, *Combinatoire des identités sur les polynômes orthogonaux*, in *Proceedings of the International Congress of Mathematicians* Vol. 2, Warsaw 1983, PWN–Polish Scientific Publications, Warsawa, and North Holland, Amsterdam, **1984**, 1541–1553.

[110] D. FOATA AND J. LABELLE, *Modèles combinatoires pour les polynômes de Meixner*, **European Journal of Combinatorics**, 4, **1983**, 305–311.

[111] D. FOATA AND P. LEROUX, *Polynômes de Jacobi, interprétation combinatoire et fonction génératrice*, **Proceedings of the American Mathematical Society**, 87, **1983**, 47–53.

[112] D. FOATA AND M.P. SCHUTZENBERGER, *Théorie géométrique des polynômes Eulériens*, Lecture Notes in Mathematics 138, Springer–Verlag, Berlin, Heidelberg, and New York, **1970**.

[113] D. FOATA AND V. STREHL, *Rearrangement of the Symmetric Group and Enumeration Properties of the Tangent and Secant Numbers*, **Mathematische Zeitschrift**, 137, **1974**, 257–264.

[114] D. FOATA AND V. STREHL, *Combinatorics of Laguerre Polynomials*, *Enumeration and Design*, editors: D.M. Jackson and S.A. Vanstone, Academic, New York, **1984**, 123–140.

[115] D. FOATA AND D. ZEILBERGER, *Laguerre Polynomials, Weighted Derangements, and Positivity*, **SIAM Journal on Discrete Mathematics**, 1, **1988**, 425–433.

[116] G.W. Ford and G.E. Uhlenbeck, *Combinatorial Problems in the Theory of Graphs III*, **Proceedings of the National Academy of Sciences USA** 42, **1956**, 529–535.

[117] J. Françon, *On the Analysis of Algorithms for Trees*, **Theoretical Computer Science**, 12, **1976**, 35–50.

[118] J. Françon, *Arbres binaires de recherche: Propriétés combinatoires et applications*, **RAIRO Informatique Théorique**, 4, **1977**, 155–169.

[119] J. Françon, *Sur le nombre de registres nécessaires à l'évaluation d'une expression arithmétique*, **RAIRO Informatique Théorique**, 18, **1984**, 355–364.

[120] J. Françon, *Fractions continues q-analogiques pour certaines familles de permutations et de partitions*, **Annales des Sciences Mathématiques du Québec**, 16, **1992**, 175–182.

[121] J. Françon and G. Viennot, *Permutations selon les pics, creux, doubles descentes, nombres d'Euler et nombres de Genocchi*, **Discrete Mathematics**, 28, **1979**, 21–35.

[122] J.F. Gagné, *Rapport existant entre la théorie des espèces et les équations différentielles*, M.Sc. Thesis, Université du Québec à Montréal, **1985**.

[123] D. Gardy, *Some Results on the Asymptotic Behaviour of Coefficients of Large Powers of Functions*, **Discrete Mathematics**, 139, **1995**, 189–217.

[124] A. Garsia and S.A. Joni, *Lecture Notes in Combinatorics*, Department of Mathematics, University of California at San Diego, La Jolla, CA, **1976**.

[125] A.M. Garsia and J. Remmel, *A Novel Form of q-Lagrange Inversion*, **Houston Journal of Mathematics**, 12, **1986**, 503–523.

[126] I. Gessel, *Generating Functions and Enumeration of Sequence*, Ph.D. Dissertation, Massachusetts Institute of Technology, Cambridge, MA, **1977**.

[127] I. Gessel, *A Combinatorial Proof of the Multivariable Lagrange Inversion Formula*, **Journal of Combinatorial Theory, Series A**, 45, **1987**, 178–195.

[128] I. Gessel, *Enumerative Applications of a Decomposition for Graphs and Digraphs*, **Discrete Mathematics**, 139, **1995**, 257–271.

[129] I. Gessel and G. Labelle, *Lagrange Inversion for Species*, **Journal of Combinatorial Theory, Series A**, 72, **1995**, 95–117.

[130] I. Gessel, B. Sagan, and Y.N. Yeh, *Enumeration of Trees by Inversions*, **Journal of Graph Theory**, 19, **1995**, 435–459.

[131] I. Gessel and D.L. Wang, *Depth-First Search as a Combinatorial Correspondence*, **Journal of Combinatorial Theory, Series A**, 26, **1979**, 308–313.

[132] I.J. Good, *Generalization to Several Variables of Lagrange's Expansion, with Applications to Stochastic Process*, **Proceedings of the Cambridge Philosophical Society**, 56, **1960**, 367–380.

[133] I.P. Goulden and D.M. Jackson, *Combinatorial Enumeration*, Wiley, New York, **1983**.

[134] I.P. GOULDEN AND D.M. JACKSON, *The Combinatorial Relationship Between Trees, Cacti and Certain Connection Coefficients for the Symmetric Group*, **European Journal of Combinatorics**, 13, **1992**, 357–365.

[135] A.O. GUELFOND, *Calcul des différences finies*, Dunod, Paris, **1963**.

[136] R.L. GRAHAM, D.E. KNUTH, AND O. PATASHNIK, *Concrete Mathematics*, Addison–Wesley, Reading, MA, **1989**.

[137] D.H. GREENE, *Labelled Formal Languages and Their Uses*, Ph.D. Dissertation, Department of Computer Science, Stanford University, Stanford CA, **1983**.

[138] W. GRÖBNER AND H. KNAPP, *Contributions to the Theory of Lie Series*, Bibliographisches Institut–Mannheim, Hochschultaschenbücher Verlag, Mannheim, Germany, **1967**.

[139] R. HAGER, A. KERBER, R. LAUE, D. MOSER, AND W. WEBER, *Construction of Orbit Representatives*, **Bayreuther Mathematische Schriften**, 35, **1991**, 157–169.

[140] P. HANLON, *A Cycle Sum Inversion Theorem*, **Journal of Combinatorial Theory, Series A**, 30, **1981**, 248–269.

[141] P. HANLON AND R.W. ROBINSON, *Counting Bridgeless Graphs*, **Journal of Combinatorial Theory, Series B**, 33, **1982**, 276–305.

[142] F. HARARY, *Graph Theory*, Addison–Wesley, Reading, MA, **1969**.

[143] F. HARARY AND R.Z. NORMAN, *The Dissimilarity Characteristic of Husimi Trees*, **Annals of Mathematics**, 58, **1953**, 134–141.

[144] F. HARARY AND E. PALMER, *Graphical Enumeration*, Academic, New York, **1973**.

[145] F. HARARY, E.M. PALMER, AND R.W. ROBINSON, *Counting Free Binary Trees Admitting a Given Height*, **Journal of Combinatorics, Information, and System Sciences**, 17, **1992**, 175–181.

[146] F. HARARY AND G. PRINS, *The Number of Homeomorphically Irreducible Trees and Other Species*, **Acta Mathematica**, 101, **1959**, 141–162.

[147] F. HARARY, G. PRINS, AND W.T. TUTTE, *The Number of Plane Trees*, **Indagaciones Math**, 26, **1964**, 319–329.

[148] F. HARARY, R.W. ROBINSON, AND A.J. SCHWENK, *Twenty-step Algorithm for Determining the Asymptotic Number of Trees of Various Species*, **Journal of the Australian Mathematical Society (Series A)**, 20, **1975**, 483–503.

[149] F. HARARY AND G.E. UHLENBECK, *On the Number of Husimi Trees*, **Proceedings of the National Academy of Sciences USA** 39, **1953**, 315–322.

[150] W.K. HAYMAN, *A Generalization of Stirling's Formula*, **Journal für die reine und angewandte Mathematik**, 196, **1956**, 67–95.

[151] P. HENNEQUIN, *Combinatorial Analysis of Quicksort Algorithm*, **RAIRO Informatique Théorique**, 23, **1989**, 317–333.

[152] J. HORNEGGER AND R. PIRASTU, *Die Speziestheorie und die Implementation iher Zykelindexreihen in Scratchpad II*, **Bayreuther Mathematische Schriften**, 43, **1993**, 11–34.

[153] A. HURWITZ, *Über einige Verallgemeinerungen der Lebniz'schen Differentiationsformel und des polynomischen Lehrsatzes*, **Zeitschrift für Mathematik und Physik**, 35, **1890**, 56–58.

[154] M. IKOLLO NDOUMBE, *Espèces asymétriques et séries indicatrices d'asymétrie*, M.Sc. Thesis, Université du Québec à Montréal, **1992**.

[155] S. JANSON, D.E. KNUTH, T. ŁUCZAK, AND B. PITTEL, *The Birth of the Giant Component*, **Random Structures and Algorithms**, 4, **1993**, 233-358.

[156] S.A. JONI, *Lagrange Inversion in Higher Dimensions and Umbral Operators*, **Linear and Multilinear Algebra**, 6, **1978**, 111–121.

[157] S.A. JONI AND G.C. ROTA, *Coalgebras and Bialgebras in Combinatorics*, **Studies in Applied Mathematics**, 61, **1979**, 93–139.

[158] A. JOYAL, *Une théorie combinatoire des séries formelles*, **Advances in Mathematics**, 42, **1981**, 1–82.

[159] A. JOYAL, *β-anneaux et vecteurs de Witt*, **Comptes rendus mathématiques de l'Académie des sciences**, La société royale du Canada, VII, **1985**, 177–182.

[160] A. JOYAL, *β-anneaux et λ-anneaux*, **Comptes rendus mathématiques de l'Académie des sciences**, La société royale du Canada, VII, **1985**, 227–232.

[161] A. JOYAL, *Règle des signes en algèbre combinatoire*, **Comptes rendus mathématiques de l'Académie des sciences**, La société royale du Canada, VII, **1985**, 285–290.

[162] A. JOYAL, *Calcul intégral combinatoire et homologie du groupe symétrique*, **Comptes rendus mathématiques de l'Académie des sciences**, La société royale du Canada, VII, **1985**, 337–342.

[163] A. JOYAL, *Foncteurs analytiques et espèces de structures*, in [196], **1986**, 126–159.

[164] A. JOYAL AND R. STREET, *The Category of Representations of the General Linear Group over a Finite Field*, **Journal of Algebra**, 176, **1995**, 908–946.

[165] A.C. KARABEG, *PQ-tree Data Structure and some Graph Embedding Problems*, Ph.D. Dissertation, University of California at San Diego, La Jolla, CA, **1988**.

[166] G.M. KELLY, *On Clubs and Doctrines*, in *Category Seminar*, Ed. G.M. Kelly, Lecture Notes in Mathematics 420, Springer–Verlag, Berlin, Heidelberg, and New York, **1974**, 181–256.

[167] R. KEMP, *A Note on the Number of Leftist Trees*, **Information Processing Letters**, 25, **1987**, 227–232.

[168] A. KERBER, *Enumeration under Finite Group Action: Symmetry Classes of Mappings*, in [196], **1986**, 160–176.

[169] A. KERBER, *La théorie combinatoire sous-tendant la théorie des représentations linéaires des groupes symétriques finis*, in *Mots*, Ed. M. Lothaire, Hermés, Paris, **1990**, 246–253.

[170] A. KERBER, *Algebraic combinatorics via finite group actions*, Bibliographisches Institut-Mannheim, Hochschultaschenbücher–Verlag, Mannheim, Germany, **1991**.

[171] A. KERBER AND K.-J. THÜRLINGS, *Counting Symmetry Classes of Functions by Weight and Automorphism Group*, Lecture Notes in Mathematics 969, Springer–Verlag, Berlin, Heidelberg, and New York, **1992**, 195–203.

[172] D.E. KNUTH, *The Art of Computer Programming*, Addison–Wesley, Reading, MA, **1981**.

[173] D.E. KNUTH AND H.S. WILF, *A Short Proof of Darboux's Lemma*, **Applied Mathematical Letters**, 2, **1989**, 139–140.

[174] D. KNUTSON, *λ-Rings and the Representations of the Symmetric Group*, Lecture Notes in Mathematics 308, Springer–Verlag, Berlin, Heidelberg, and New York, **1973**.

[175] G. KREWERAS, *Une famille de polynômes ayant plusieurs propriétés énumératives*, **Periodica Math. Hungarica**, 11, **1980**, 309–320.

[176] M. KUCZMA, B. CHOCZEWSKI, AND R. GER, *Iterative Functional Equations*, *Encyclopedia of Mathematics and its Applications* 32, Cambridge University Press, New York, **1990**.

[177] G. LABELLE, *Sur l'inversion et l'itération continue des séries formelles*, **European Journal of Combinatorics**, 1, **1980**, 113–138.

[178] G. LABELLE, *Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange*, **Advances in Mathematics**, 42, **1981**, 217–247.

[179] G. LABELLE, *Une combinatoire sous-jacente au théorème des fonctions implicites*, **Journal of Combinatorial Theory, Series A**, 40, **1985**, 377–393.

[180] G. LABELLE, *Eclosions combinatoires appliquées à l'inversion multidimensionnelle des séries formelles*, **Journal of Combinatorial Theory, Series A**, 39, **1985**, 52–82.

[181] G. LABELLE, *Some New Computational Methods in the Theory of Species*, in [196], **1986**, 192–209.

[182] G. LABELLE, *Interpolation dans les κ-espèces*, in *Actes du séminaire lotharingien de combinatoire, 14e session*, Ed. V. Strehl, Publications de l'Institut de recherche mathématiques, Strasbourg, France, **1986**, 60–70.

[183] G. LABELLE, *On Combinatorial Differential Equations*, **Journal of Mathematical Analysis Applications**, 113, **1986**, 344–381.

[184] G. LABELLE, *Dérivées directionnelles et développements de Taylor combinatoires*, **Discrete Mathematics**, 79, **1989**, 279–297.

[185] G. LABELLE, *On the Generalized Iterates of Yeh's Combinatorial κ-Species*, **Journal of Combinatorial Theory, Series A**, 50, **1989**, 235–258.

[186] G. LABELLE, *On Asymmetric Structures*, **Discrete Mathematics**, 99, **1992**, 141–164.

[187] G. LABELLE, *Counting Asymmetric Enriched Trees*, **Journal of Symbolic Computation**, 14, **1992**, 211–242.

[188] G. Labelle, *Sur la symétrie et l'asymétrie des structures combinatoires*, **Theoretical Computer Science**, 117, **1993**, 3–22.

[189] G. Labelle, *Some Combinatorial Results First Found Using Computer Algebra*, **Journal of Symbolic Computation**, 20, **1995**, 567–594.

[190] G. Labelle, J. Labelle, and K. Pineau, *Sur une généralisation des séries indicatrices d'espèces*, **Journal of Combinatorial Theory, Series A**, 19, **1995**, 17–35.

[191] G. Labelle and L. Laforest, *Sur la distribution de l'arité de la racine d'une arborescence hyperquaternaire à d dimensions*, **Discrete Mathematics**, 139, **1995**, 287–302.

[192] G. Labelle and L. Laforest, *Combinatorial Variations on Multidimensional Quadtrees*, **Journal of Combinatorial Theory, Series A**, 69, **1995**, 1–16.

[193] G. Labelle and L. Laforest, *Etude de constantes universelles pour les arborescences hyperquaternaires de recherche*, **Discrete Mathematics**, 153, **1996**, 199–211.

[194] G. Labelle and P. Leroux, *Enumeration of (Uni- or Bicolored) Plane Trees According to their Degree Distribution*, **Discrete Mathematics**, 157, **1996**, 227–240.

[195] G. Labelle and P. Leroux, *Extension of the Exponential Formula in Enumerative Combinatorics*, **Electronic Journal of Combinatorics**, 3 (2), **1996**, # R12, 14 p.

[196] G. Labelle and P. Leroux, editors, **Combinatoire Énumérative**, Lecture Notes in Mathematics 1234, Springer–Verlag, Berlin, Heidelberg, and New York, **1986**.

[197] G. Labelle and K. Pineau, *Ensembles orientés et permutations paires: séries indicatrices d'asymétrie et q-séries*, **Advances in Applied Mathematics**, 15, **1994**, 452–475.

[198] J. Labelle, *Théorie des Graphes*, Modulo, Montréal, **1981**.

[199] J. Labelle, *Applications diverses de la théorie combinatoire des espèces de structures*, **Annales des Sciences Mathématiques du Québec**, 7, **1983**, 59–94.

[200] J. Labelle, *Quelques espèces sur les ensembles de petite cardinalité*, **Annales des Sciences Mathématiques du Québec**, 11, **1985**, 31–58.

[201] J. Labelle, *Langages de Dyck généralisés*, **Annales des Sciences Mathématiques du Québec**, 17, **1993**, 1–13.

[202] J. Labelle and Y.N. Yeh, *A Combinatorial Model for Hahn Polynomials*, in *Actes du séminaire lotharingien de combinatoire, 16e session*, Ed. D. Foata, Publications de l'Institut de recherche mathématiques, Strasbourg, France, **1988**, 99–107.

[203] J. Labelle and Y.N. Yeh, *Some Combinatorics of the Hypergeometric Series*, **European Journal of Combinatorics**, 9, **1988**, 593–605.

[204] J. Labelle and Y.N. Yeh, *The Combinatorics of Laguerre, Charlier and Hermite Polynomials*, **Studies in Applied Mathematics**, 80, **1989**, 25–36.

[205] J. Labelle and Y.N. Yeh, *The Relation Between Burnside Rings and Combinatorial Species*, **Journal of Combinatorial Theory, Series A**, 50, **1989**, 269–284.

[206] J. Labelle and Y.N. Yeh, *Combinatorial Proofs of Symmetry Formulas for the Generalized Hypergeometric Series*, **Journal of Mathematical Analysis Applications**, 139, **1989**, 36–48.

[207] J. Labelle and Y.N. Yeh, *Dyck Paths of Knight Moves*, **Discrete Applied Mathematics**, 24, **1989**, 213–221.

[208] J. Labelle and Y.N. Yeh, *Combinatorial Proofs of some Limit Formulas Involving Orthogonal Polynomials*, **Discrete Mathematics**, 79, **1989-1990**, 77–93.

[209] J. Labelle and Y.N. Yeh, *Generalized Dyck Paths*, **Discrete Mathematics**, 82, **1990**, 1–6.

[210] L. Laforest, *Etude des arbres Hyperquaternaires*, Ph.D. Dissertation, Université McGill, **Publications du Laboratoire de combinatoire et d'informatique mathématique** Vol. 3, Université du Québec à Montréal, **1990**.

[211] F. Lamnabhi–Lagarrigue, P. Leroux, and X.G. Viennot, *Combinatorial Approximations of Volterra Series by Bilinear Systems*, in *Analysis of Controlled Dynamical Systems*, editors: B. Bonnard, B. Bride, J.-P. Gauthier, and I. Gupka, Progress in Systems and Control Theory 8, Birkhäusser, Boston, **1991**, 304–315.

[212] P. Leroux, *The Isomorphism Problem for Incidence Algebras of Möbius Categories*, **Illinois Journal of Mathematics**, 26, **1982**, 52–61.

[213] P. Leroux, *Methoden der Anzahlbestimmung für Einige Klassen von Graphen*, **Bayreuther Mathematische Schriften**, 26, **1988**, 1–36.

[214] P. Leroux, *Reduced Matrices and $q-\log$ Concavity Properties of q-Stirling Numbers*, **Journal of Combinatorial Theory, Series A**, 54, **1990**, 64–84.

[215] P. Leroux and B. Miloudi, *Généralisations de la formule d'Otter*, **Annales des Sciences Mathématiques du Québec**, 16, **1992**, 53–80.

[216] P. Leroux and V. Strehl, *Jacobi Polynomials: Combinatorics of the Basic Identities*, **Discrete Mathematics**, 57, **1985**, 167–187.

[217] P. Leroux and G.X. Viennot, *Combinatorial Resolution of Systems of Differential Equations I: Ordinary Differential Equations*, in [196], **1986**, 233–253.

[218] P. Leroux and G.X. Viennot, *Résolution combinatoire des équations différentielles II: Calcul intégral combinatoire*, **Annales des Sciences Mathématiques du Québec**, 12, **1988**, 210–245.

[219] P. Leroux and G.X. Viennot, *Combinatorial Resolution of Systems of Differential Equations IV: Separation of Variables*, **Discrete Mathematics**, 72, **1988**, 237–250.

[220] P. Leroux and G.X. Viennot, *A Combinatorial Approach to Non Linear Functional Expansions: An Introduction With an Example*, **Theoretical Computer Science**, 79, **1991**, 179–193.

[221] J.-B. Lévesque, *Sur la pondération en combinatoire énumérative*, M.Sc. Thesis, Université du Québec à Montréal, **1989**.

[222] C.I. LIU, *Introduction to Combinatorial Mathematics*, McGraw–Hill, New York, **1968**.

[223] V.A. LISKOVETS AND T.R.WALSH, *Ten Steps to Counting Planar Graphs*, **Congressus Numerantium**, 60, **1987**, 269–277.

[224] J.-L. LODAY, *La renaissance des opérades*, **Séminaire Bourbaki**, 792, **1994–95**.

[225] D.E. LOEB, *Sequences of Symmetric Functions of Binomial Type*, **Studies in Applied Mathematics**, 83, **1990**, 1–30.

[226] A. LONGTIN, *Une combinatoire non-commutative pour l'étude des nombres sécants*, in [196], **1986**, 246–266.

[227] M. LOTHAIRE, *Combinatorics on Words*, *Encyclopedia of Mathematics* Vol. 17, Addison–Wesley, Reading, MA, **1983**.

[228] I.G. MACDONALD, *Symmetric Functions and Hall Polynomials*, 2nd edition, Clarendon, Oxford, **1995**.

[229] S. MACLANE, *Categories for the Working Mathematician*, Springer–Verlag, Berlin, Heidelberg, and New York, **1971**.

[230] S. MACLANE, *Coherence in Categories*, Lecture Notes in Mathematics 281, Springer–Verlag, Berlin, Heidelberg, and New York, **1972**.

[231] P.A. MACMAHON, *Combinatorial Analysis*, 2 volumes, Cambridge University Press, London, 1915–1916. Reprinted: Chelsea, New York, **1960**.

[232] C.L. MALLOWS AND J. RIORDAN, *The Inversion Enumerator for Labeled Trees*, **Bulletin of the American Mathematical Society**, 74, **1968**, 92–94.

[233] J.P. MAY, *The Geometry of Iterated Loops*, Lecture Notes in Mathematics 271, Springer–Verlag, Berlin, Heidelberg, and New York, **1972**.

[234] A. MEIR AND J.W. MOON, *On the Altitude of Nodes in Random Trees* **Canadian Journal of Mathematics**, 30, **1978**, 997–1015.

[235] A. MEIR AND J.W. MOON, *On the Branch Sizes of Rooted Unlabeled Trees*, in *Graph Theory and its Applications: East and West*, Jinan 1986, **Annals of the New York Academy of Sciences**, 576, **1989**, 399–407.

[236] A. MEIR AND J.W. MOON, *Some Asymptotic Results Useful in Enumerative Problems*, **Aequationes Math.**, 33, **1987**, 260–268.

[237] A. MEIR AND J.W. MOON, *On Nodes of Out-Degree One in Random Trees*, **Combinatorics**, 52, **1987**, 405–416.

[238] A. MEIR AND J.W. MOON, *Recursive Trees with no Nodes of Out-Degree One*, **Congressus Numerantium**, 66, **1988**, 49–62.

[239] A. MEIR AND J.W. MOON, *On an Asymptotic Method in Enumeration*, **Journal of Combinatorial Theory, Series A**, 51, **1989**, 77–89.

[240] A. MEIR AND J.W. MOON, *On the Maximum Out-Degree in Random Trees*, **Austirlian Journal of Combinatorics**, 2, **1990**, 147–156.

[241] A. MEIR AND J.W. MOON, *The Asymptotic Behavior of Coefficients of Powers of Certain Generating Functions*, **European Journal of Combinatorics**, 11, **1990**, 581–587.

[242] A. MEIR AND J.W. MOON, *On Nodes of Large Out-Degree in Random Trees*, **Congressus Numerantium**, 82, **1991**, 3–13.

[243] A. MEIR, J.W. MOON, AND J. MYCIELSKY, *Hereditary Finite Sets and Identity Trees*, **Journal of Combinatorial Theory, Series B**, 35, **1983**, 142–155.

[244] K. MEHLHORN, *Data Structures and Algorithms, I: Sorting and Searching*, Springer–Verlag, Berlin, Heidelberg, and New York, **1985**.

[245] M. MÉNDEZ, *Tensor Species and Symmetric Functions*, **Proceedings of the National Academy of Sciences USA** , 88, **1991**, 9892–9894.

[246] M. MÉNDEZ, *Multisets and the Combinatorics of Symmetric Functions*, **Advances in Mathematics**, 102, **1993**, 95–125.

[247] M. MÉNDEZ AND O. NAVA, *Colored Species, c-Monoids and Plethysm*, **Journal of Combinatorial Theory, Series A**, 64, **1993**, 102–129.

[248] M. MÉNDEZ AND J. YANG, *Möbius Species*, **Advances in Mathematics**, 85, **1991**, 83–128.

[249] N.C. METROPOLIS AND G.C. ROTA, *Witt Vectors and the Algebra of Necklaces*, **Advances in Mathematics**, 50, **1983**, 95–125.

[250] N.C. METROPOLIS AND G.C. ROTA, *The Cyclotomic Identity*, in *Combinatorics and Algebra*, **Contemporary Mathematics**, American Mathematical Society, 34, **1984**, 19–28.

[251] J.W. MOON, *Counting Labelled Trees*, **Canadian Mathematical Monographs** 1, Canadian Mathematical Society, **1970**.

[252] J.W. MOON, *Some Enumerative Results on Series Parallel Networks*, **Annals of Discrete Mathematics**, 33, **1987**, 199–226.

[253] R. MULLIN AND G.C. ROTA, *On the Foundation of Combinatorial Theory III: Theory of Binomial Enumeration*, in *Graph Theory and its Applications*, Ed. B. Harris, Academic, New York, **1970**, 167–213.

[254] A. NICKEL, *Abzählung und Konstruction diskreter und die Theorie des Spezies*, Diplomarbeit, Universität Bayreuth, Bayreuth, Germany, **1989**.

[255] O. NAVA, *On the Combinatorics of Plethysm*, **Journal of Combinatorial Theory, Series A**, 46, **1987**, 215–251.

[256] O. NAVA AND G.C. ROTA, *Plethysm, Categories and Combinatorics*, **Advances in Mathematics**, 58, **1985**, 61–68.

[257] R.Z. NORMAN, *On the Number of Linear Graphs With Given Blocks*, Ph.D. Dissertation, University of Michigan, Ann Arbor, **1954**.

[258] A. ODLYZKO, *Periodic Oscillations of Power Series that Satisfy Functional Equations*, **Advances in Mathematics**, 44, **1982**, 180–225.

[259] R. OTTER, *The Number of Trees*, **Annals of Mathematics**, 49, **1948**, 583–599.

[260] T. Ottmann and D. Wood, *1-2-Brother Trees or AVL Trees Revisited*, **Computer Journal**, 23, **1980**, 248–255.

[261] J.J. Pansiot, *Nombres d'Euler et inversions dans les arbres*, **European Journal of Combinatorics**, 3, **1982**, 259–262.

[262] K. Pineau, *Une généralisation des séries indicatrices d'espèces de structures*, Ph.D. Dissertation, Université du Québec à Montréal, **Publications du Laboratoire de combinatoire et d'informatique mathématique** Vol. 21, **1995**.

[263] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und Chemische Verbindungen*, **Acta Mathematica**, 68, **1937**, 145–254.

[264] G. Pólya and R.C. Read, *Combinatorial Enumeration of Groups, Graphs and Chemical Compounds*, Springer–Verlag, Berlin, Heidelberg, and New York, **1987**.

[265] G. Pólya and G. Szego, *Problems and Theorems in Analysis*, Springer–Verlag, Berlin, Heidelberg, and New York, **1976**.

[266] M. Poulin, *Relations entre certaines fonctions arithmétiques et séries indicatrices en combinatoire*, M.Sc. Thesis, Université du Québec à Montréal, **1992**.

[267] D.S. Rajan, *The Adjoints of the Derivative Functor on Species*, **Journal of Combinatorial Theory, Series A**, 62, **1993**, 93–106.

[268] D.S. Rajan, *The Equations $D^kY=X^n$ in Combinatorial Species*, **Discrete Mathematics**, 118, **1993**, 197–206.

[269] M.V.S. Ramanath and T.R. Walsh, *Enumeration and Generation of a Class of Regular Digraphs*, **Journal of Graph Theory**, 11, **1987**, 471–479.

[270] D. Rawlings, *The ABC's of Classical Enumeration*, **Annales des Sciences Mathématiques du Québec**, 10, **1986**, 207–235.

[271] D. Rawlings, *A Binary Tree Decomposition Space of Permutation Statistics*, **Journal of Combinatorial Theory, Series A**, 59, **1992**, 111–124.

[272] N. Ray, *Umbral Calculus, Binomial Enumeration and Chromatic Polynomials*, **Transactions of the American Mathematical Society**, 309, **1988**, 191–213.

[273] A.H. Read, *The Solution of a Functional Equation*, **Proceedings of the Royal Society**, Edinburgh, A 63, **1951–1952**, 336–345.

[274] R.C. Read, *The Use of S-Functions in Combinatorial Analysis*, **Canadian Journal of Mathematics**, 20, **1968**, 808–841.

[275] J.M. Redfield, *The Theory of Group-Reduced Distributions*, **American Journal of Mathematics**, 49, **1927**, 433–455.

[276] C. Reutenauer, *Séries formelles et algèbres syntactiques*, **Journal of Algebra**, 66, **1980**, 448–483.

[277] C. Reutenauer, *Propriétes arithmétiques de séries rationnelles et ensembles denses*, **Acta Arithmetica**, 39, **1981**, 133–144.

[278] C. Reutenauer, *Theorem of Poincaré-Birkhoff-Witt, Logarithm and Representations of the Symmetric Group Whose orders are the Stirling Numbers*, in [196], **1986**, 267–293.

[279] C. Reutenauer, *Mots circulaires et polynômes irréductibles*, **Annales des Sciences Mathématiques du Québec**, 12, **1988**, 275–285.

[280] C. Reutenauer, *Free Lie Algebras*, Oxford University Press, Oxford, **1993**.

[281] J. Riordan, *An Introduction to Combinatorial Analysis*, Princeton University Press, Princeton, NJ, **1980**.

[282] R.W. Robinson, *Enumeration of Non-Separable Graphs*, **Journal of Combinatorial Theory, Series B**, 9, **1970**, 327–356.

[283] R.W. Robinson and T.R. Walsh, *Inversion of Cycle Index Series Relations for 2- and 3-Connected Graphs*, **Journal of Combinatorial Theory, Series B**, 57, **1993**, 289–308.

[284] G.C. Rota, *On the Foundations of Combinatorial Theory, I: Theory of Möbius Functions*, **Zeitschift für Wahrscheinlichkeitstheorie und Verwandte Gebiete**, 2, **1964**, 340–368.

[285] G.C. Rota, *Baxter Algebras and Combinatorial Identities, II*, **Bulletin of the American Mathematical Society**, 75, **1969**, 330–334.

[286] G.C. Rota, *Finite Operator Calculus*, Academic, New York, **1975**.

[287] G.C. Rota, *Hopf Algebras in Combinatorics*, in *Problèmes combinatoires et théorie des graphes*, **Colloques internationaux du CNRS**, 260, Paris, **1978**, 363–365.

[288] G.C. Rota and B.E. Sagan, *Congruences Derived from Group Action*, **European Journal of Combinatorics**, 1, **1980**, 67–76.

[289] G.C. Rota and D.A. Smith, *Enumeration Under Group Action*, **Annali Scuola Normale Superiore Pisa**, Classe di Scienze, Serie IV, Vol. IV, **1977**, 637–646.

[290] B. Sagan, *The Symmetric Group*, Wadsworth Brooks/Cole, Pacific Grove CA, **1991**.

[291] B. Salvy, *Asymptotique, automatique et fonctions génératrices*, Ph.D. Dissertation, Ecole Polytechnique, Palaiseau, France, **1991**.

[292] W.R. Schmitt, *Hopf Algebras of Combinatorial Structures*, **Canadian Journal of Mathematics**, 45, **1993**, 412–428.

[293] M.-P. Schützenberger, *Solutions non commutatives d'une équation différentielle classique*, in *New Concepts and Technologies in Parallel Information Processing*, Ed. M. Caianello, *NATO Advanced Studies Institute Series E: Applied Sciences*, 9, **1975**, 381–401.

[294] H.I. Scoins, *The Number of Trees With Nodes of Alternate Parity*, **Proceedings of the Cambridge Philosophical Society**, 58, **1962**, 12–16.

[295] R. Sedgewick and P. Flajolet, *An Introduction to the Analysis of Algorithms*, Addison–Wesley, Reading, MA, **1996**.

[296] D. Senato and A.M. Venezia, *Polynomial Species and Connections Among Bases of the Symmetric Polynomials*, *Combinatorics'86*, Trento, Annals of Discrete Mathematics 37, **1988**, 405–412.

[297] J.P. SERRE, *Linear Representation of Finite Groups*, Springer–Verlag, Berlin, Heidelberg, and New York, **1977**.

[298] N.J.A. SLOANE, *An On-Line Version of the Encyclopedia of Integer Sequences*, **Electronic Journal of Combinatorics**, 1 (1), **1994**.

[299] N.J.A. SLOANE AND S. PLOUFFE, *The Encyclopedia of Integer Sequences*, Academic, New York, **1995**.

[300] R.P. STANLEY, *Binomial Posets, Möbius Inversion, and Permutation Enumeration*, **Journal of Combinatorial Theory, Series A**, 20, **1976**, 336–356.

[301] R.P. STANLEY, *Exponential Structures*, **Studies in Applied Mathematics**, 59, **1978**, 73–82.

[302] R.P. STANLEY, *Generating Functions*, in *MAA Studies in Combinatorics*, Ed. G.C. Rota, Mathematical Association of America, Washington D.C., **1978**, 100–141.

[303] R.P. STANLEY, *Differentiably Finite Power Series*, **European Journal of Combinatorics**, 1, **1980**, 175–188.

[304] R.P. STANLEY, *Enumerative Combinatorics*, Volume 1, Wadsworth Brooks/Cole, **1986**.

[305] D. STANTON, *Recent Results for the q-Lagrange Inversion Formula*, *Conference on Ramanujan revisited*, Urbana-Champaign, Ill., 1987, Academic, New York, Boston, **1988**, 525–536.

[306] D. STANTON AND D. WHITE, *Constructive Combinatorics*, Undergraduate Texts in Mathematics, Springer–Verlag, Berlin, Heidelberg, and New York, **1986**.

[307] P.K. STOCKMEYER, *Enumeration of Graphs with Prescribed Automorphism Group*, Ph.D. Dissertation, University of Michigan, Ann Arbor, **1971**.

[308] V. STREHL, *Combinatorics of Jacobi-configurations I: Complete Oriented Matching*, in [196], **1986**, 294–307.

[309] V. STREHL, *Combinatorics of Jacobi-configurations II: A Rational Approximation via Matching Polynomials*, in *Actes du séminaire lotharingien de combinatoire, 13e session*, Ed. G. Nicoletti, Publications del'Institut de recherche mathématiques, Strasbourg, France, **1986**, 112–133.

[310] V. STREHL, *Combinatorics of Jacobi-Configurations III: The Srivastava–Singhal Generating Relation Revisited*, **Discrete Mathematics**, 73, **1988**, 221–232.

[311] V. STREHL, *Zykel-Enumeration bei Lokal-Strukturierten Funktionen*, Habilitationsschrift, Institut für Mathematische Maschinen und Datenverarbeitung der Universität Erlangen-Nürnberg, Germany, **1989**.

[312] V. STREHL, *Cycle Counting for Isomorphism Types of Endofunctions*, **Bayreuther Mathematische Schriften**, 40, **1992**, 153–167.

[313] W.T. TUTTE, *Connectivity in Graphs*, University of Toronto Press, Toronto, **1966**.

[314] D.E. TAYLOR, *A Natural Proof of the Cyclotomic Identity*, **Bulletin of the Australian Mathematical Society**, 42, **1990**, 185–189.

[315] B. UNGER, *Asymptotics of Increasing Trees*, Ph.D. Dissertation, University of Sydney, **1993**.

[316] J.H. VAN LINT AND R.M. WILSON, *A Course in Combinatorics*, Cambridge University Press, Cambridge, **1992**.

[317] G.X. VIENNOT, *Une interprétation combinatoire des coefficients du développement en série entière des fonctions elliptiques de Jacobi*, **Journal of Combinatorial Theory, Series A**, 29, **1980**, 121–133.

[318] G.X. VIENNOT, *Interprétation combinatoire des nombres d'Euler et de Genocchi*, **Séminaire de théorie des nombres** 11, Université de Bordeaux, Bordeaux, France, **1980–1981**.

[319] G.X. VIENNOT, *Une théorie combinatoire des polynômes orthogonaux*, **Publications du Laboratoire de combinatoire et d'informatique mathématique**, Université du Québec à Montréal, **1983**.

[320] G.X. VIENNOT, *Heaps of Pieces I: Basic Definitions and Combinatorial Lemmas*, in [196], **1986**, 321–350.

[321] J. VUILLEMIN, *A Unifying Look at Data Structures*, **Communications of the ACM**, 23, **1980**, 229–239.

[322] D.W. WALKUP, *The Number of Plane Trees*, **Mathematika**, 19, **1972**, 200–204.

[323] T.R. WALSH, *On the Size of Quadtrees Generalized to d-Dimensional Binary Pictures*, **Computers and Mathematics with Applications**, 11, **1985**, 1089–1097.

[324] K.H. WEHRHAHN, *Combinatorics, an Introduction*, 2nd edition, Carslaw Publications, Glebe, NSW, Australia, **1992**.

[325] D. WHITE, *Counting Patterns With a Given Automorphism Group*, **Proceedings of the American Mathematical Society**, 47, **1975**, 41–44.

[326] H.S. WILF, *Generatingfunctionology*. Academic, New York, **1990**.

[327] H.S. WILF AND D. ZEILBERGER, *Towards Computerized Proofs of Identities*, **Bulletin of the American Mathematical Society**, New Series 23, **1990**, 77–83.

[328] S. WOLFRAM, *Mathematica, A System for Doing Mathematics by Computer*, Addison–Wesley, Reading, MA, **1988**.

[329] S.G. WILLIAMSON, *Combinatorics for Computer Science*, Computer Science Press, Rockville, Maryland, **1985**.

[330] N.WORWALD AND E.M. WRIGHT, *The Exponential Generating Functon of Labelled Blocks*, **Discrete Mathematics**, 25, **1979**, 93–96.

[331] J.S. YANG, *Asymmetric Structures, Types and Indicator Polynomials*, **Advances in Mathematics**, 100, **1993**, 133–153.

[332] J.S. YANG, *The Plethystic Inverse of a Formal Power Series*, **Discrete Mathematics**, 139, **1995**, 413–442.

[333] Y.N. YEH, *On the Combinatorial Species of Joyal*, Ph.D. Dissertation, State University of New York at Buffalo, **1985**.

[334] Y.N. YEH, *The Calculus of Virtual Species and $\mathbb{K}$-Species*, in [196], **1986**, 351–369.

[335] Y.N. YEH, *Solutions to Some Problems in the Theory of Combinatorial Species*, **Annales des Sciences Mathématiques du Québec**, 11, **1987**, 363–369.

[336] D. ZEILBERGER, *A Combinatorial Approach to Matrix Algebra*, **Discrete Mathematics**, 56, **1985**, 61–72.

[337] D. ZEILBERGER, *Toward a Combinatorial Proof of the Jacobian Conjecture?*, in [196], **1986**, 370–380.

[338] D. ZEILBERGER, *A Bijection From Ordered Trees to Binary Trees that Sends the Pruning Order to the Strahler Number*, **Discrete Mathematics**, 82, **1990**, 89–92.

[339] J. ZENG, *La β-extension de la formule d'inversion de Lagrange à plusieurs variables*, in *Actes du séminaire lotharingien de combinatoire, 20e session*, editors: L. Cerlienco and D. Foata, **Publications de l'Institut de recherche mathématiques**, Strasbourg, France, **1988**, 23–38.

[340] J. ZENG, *Weighted Derangements and the Linearization Coefficients of Orthogonal Sheffer Plolynomials*, **Proceedings of the London Mathematical Society**, 65, **1992**, 1–22.

Index