COVERING SPACES

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Abstract. We will study the concept of the fundamental group of a topological space. In addition, we will also study covering spaces of a topological space and its relation with the fundamental group. At the end we will highlight the analogy between covering spaces and the Galois group which will be evident from the theory we build on covering spaces.

Contents

1. Introduction

While studying topology, the first question we ask is, whether two spaces are "topologically" equivalent. If there exists a homeomorphism between two topological spaces, we can show that they have the same properties. However, finding homeomorphisms or proving that a homeomorphism does not exist between two spaces is a difficult task. For example, how do we prove that $\mathbb{R}^2$ and $\mathbb{R}^2 \backslash (0, 0)$ are not topologically the same? This is where the concept of "The Fundamental Group" comes in. Fundamental groups give an algebraic structure to topology. If two spaces are topologically equivalent, then the spaces have isomorphic fundamental groups.

Suppose X is a topological space. A covering of X is a space $\tilde{X}$ with a continuous map onto X, that satisfies a very strong condition. We will see the precise definition in Section 4. As a first example, think of "wrapping" the real line around a circle.

The fundamental group of a space is closely related to its covering space. The subgroups of the fundamental group of a topological space X can help us classify all the covering spaces of X.

Recall that the fundamental theorem of Galois theory states that for each intermediate subfield we have a corresponding subgroup of the Galois group. This is analogous to the one to one correspondence between the subgroups of the fundamental group of the space X and its covering spaces in algebraic topology. We will highlight this analogy at the end of exploring covering spaces and its relation with the fundamental group.

The purpose of this paper is to report on our study of the topic of covering spaces. This study was a part of the Apprenticeship program of the Department of Mathematics, University of Chicago REU of 2017. We follow the exposition in Massey[1]. This paper is organised as follows. In Section 2 we explain the concept of fundamental groups. In Section 3 we review some group theoretic concepts before explaining the idea of covering spaces. We conclude by discussing the analogy with the Galois correspondence.

2. Fundamental groups

2.1. **Homotopy.** Let us begin with some of the basic definitions we will need to study fundamental groups. At all times I denotes the unit interval $[0, 1]$.

Definition 2.1 (Homotopy)**.** A homotopy is a family of maps $f_t : X \to Y, t \in I$ such that the associated map $F : X \times I \to Y$ given by $F(x, t) = f_t(x)$ is continuous. Two maps f and g are homotopic if there exists a homotopy f_t between them. We write $f \simeq g$.

A homotopy from X to Y gives us a continuous way of deforming X into Y.

Definition 2.2 (Homotopy equivalence)**.** Let $f : X \to Y$ be a continuous map. We say that f is a homotopy equivalence if there exists a continuous map $g : Y \to X$ such that $f \circ g$ is homotopic to the identity map of Y and $g \circ f$ is homotopic to the identity map of X. In such a case we say that X and Y are homotopy equivalent.

A special case of homotopy is deformation retraction. Deformation retraction of a space X onto a subspace A is a family of continuous maps $f_t : X \to X, t \in [0, 1]$ such that f_0 is the identity map (we will denote $\mathbb{I}$ as the identity map) and $f_1(X) \subset A$. Also, $f_t|_A = \mathbb{I}$.

Geometrically, think of it this way: We draw line segments from space X onto the subspace A and then let X shrink into A along the line segments. For example, S^1 is a deformation retract of $\mathbb{R}^2 \backslash (0, 0)$.

Remark 2.3. Let X be a topological space and A be a subspace of X. If A is a deformation retract of X, then X is homotopy equivalent to A.

Definition 2.4 (Path homotopy)**.** Let X be a topological space. Two paths f and f' in X with the same initial point x_0 and terminal point x_1 are path homotopic if there is a continuous map $F : I \times I \to X$ such that:

$$F(s,0) = f(s)\ ; F(s,1) = f'(s) \tag{2.5}$$

$$F(0,t) = x_0\ ; F(1,t) = x_1,\ \text{for all } s,t \in I. \tag{2.6}$$

Here F is a path homotopy between f and f'. We write $f \simeq_p f'$. The relation (2.5) says that F represents a continuous way of deforming f into f' and (2.6) says that the end points of the paths stay fixed. Note that f is always (path) homotopic to itself with $F(s,t) = f(s)$ for all t.

Lemma 2.7. *Homotopy and path homotopy are equivalence relations.*

Proof. Reflexivity of the relation is clear. Let us look at how the relation is symmetric and transitive. Suppose $f \simeq f'$ and F is the homotopy, then notice that $G(x,t) = F(x, 1-t)$ is the homotopy between f' and f. Thus the relation is symmetric as well.

Now, suppose $f \simeq f'$ and $f' \simeq f''$. Let F be the homotopy between f and f' and F' be the homotopy between f' and f''. Define $G : X \times I \to X$ as follows:

$$G(x,t) = \begin{cases} F(x,2t) & t \in \left[0, \frac{1}{2}\right] \\ F'(x,2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

This is a well defined homotopy between f and f''. The motivation behind defining the map this way was to first deform f into f' and then f' into f'' in half the time. ☐

Note that any two paths f_0 and f_1 in $\mathbb{R}^n$ having the same end points x_0 and x_1 are homotopic via the linear homotopy $F(x,t) = f_t(x) = (1-t)f_0(x) + tf_1(x)$. During this homotopy each point of $f_0(x)$ travels along the line segment joining it to $f_1(x)$.

2.2. **Product of paths.**

Definition 2.8. Given two paths $f, g : I \to X$ such that $f(1) = g(0)$, we define the product of f and g as follows:

$$f \cdot g(s) = \begin{cases} f(2s) & s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

From the definition it is clear that the product of two paths f and g simply means that we traverse f first and then g in half the time. This product preserves homotopy. Also note that if we restrict to paths with the same starting and terminal point (say) x_0, then we have a loop. We will refer to x_0 as the base point.

Notation 2.9. The set of all homotopy classes $[f]$ of loops $f : S^1 \to X$ is denoted by $\pi_1(X, x_0)$.

Remark 2.10. We can also consider closed paths from $S^n \to X$ and have higher order homotopy classes. (The set is then denoted by $\pi_n(X, x_0)$). However, in this paper we will only consider $\pi_1(X, x_0)$. For simplicity, we drop the subscript.

2.3. **The Fundamental Group.**

Proposition 2.11. *$\pi(X, x_0)$ is a group with respect to $[f][g] = [f \cdot g]$.*

Note that since the product of paths preserves homotopy, the above is well defined on homotopy classes. The result follows from the following lemmas we prove to show that the group axioms hold for $\pi(X, x_0)$.

Lemma 2.12. *The multiplication of equivalence classes of paths is associative.*

Proof. Let f, g, h be paths such that terminal point of f is the initial point of g and the terminal point of g is the initial point h. We need to show that:

$$(f \cdot g) \cdot h \simeq f \cdot (g \cdot h).$$

We will give a pictorial proof of the above lemma. Consider Figure 1. Here at $t = 0$, we have $(f \cdot g) \cdot h$ and at $t = 1$, the interval is "deformed" into $f \cdot (g \cdot h)$ along the two line segments. □

For any $x \in X$ denote $[\epsilon_x]$, the equivalence class of the constant map from I into the point x in X.

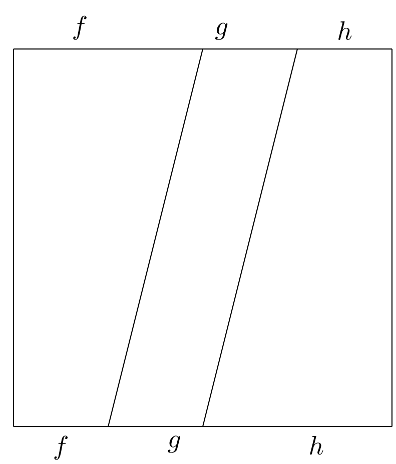


FIGURE 1

Lemma 2.13. *Let $[f]$ be an equivalence class of paths with the initial point x and the terminal point y. Then, $[\epsilon_x][f] = [f]$ and $[f][\epsilon_y] = [f]$.*

As in the above proof, we can argue that $\epsilon_x \cdot f \simeq f$ and $f \simeq \epsilon_y \cdot f$ (see Figure 2). Note that in case of a loop, the left identity is same as the right identity.

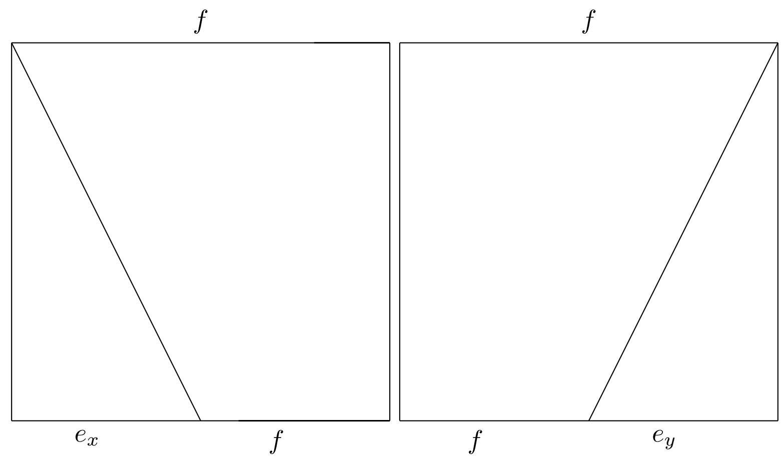


FIGURE 2

Lemma 2.14. *Let f be a path in X and let $\hat{f}$ be the path $f(1-t), t \in I$, i.e., the path f in the opposite direction. Let $[f]$ and $[\hat{f}]$ denote the equivalence classes of f and $\hat{f}$ respectively. Then, $[f][\hat{f}] = [\epsilon_x]$ and $[\hat{f}][f] = [\epsilon_y]$.*

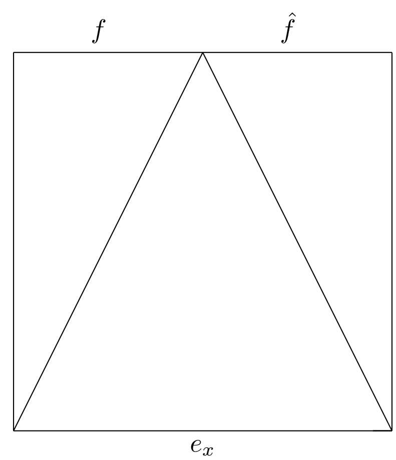


FIGURE 3

Proof. The diagram shows that $f \cdot \hat{f} \simeq \epsilon$ where ϵ is the constant path at $x \in X$. □

From the above three lemmas, it is clear that $\pi(X, x_0)$ forms a group. We call $\pi(X, x_0)$, "The Fundamental Group" of the space X.

Remark 2.15. The reader should note here that the product of paths is not associative, but the product of equivalence classes, under the relation of homotopy, is associative.

How does the fundamental group depend on the choice of the base point? The following theorem tells us that we don't have to worry about what base point to choose while considering the fundamental group of a path connected space.

Theorem 2.16. *If X is path connected then the groups $\pi(X, x)$ and $\pi(Y, y)$ are isomorphic for any points $x, y \in X$.*

Proof. Let $x, y \in X$ and let γ be a path class with initial point x and terminal point y. Define a map $u : \pi(X, x) \to \pi(X, y)$ such that $\alpha \mapsto \gamma^{-1}\alpha\gamma$. Note that:

$$\begin{aligned} u(\alpha\beta) &= \gamma^{-1}\alpha\beta\gamma \\ &= (\gamma^{-1}\alpha\gamma) \cdot (\gamma^{-1}\beta\gamma) \\ &= u(\alpha) \cdot u(\beta) \end{aligned}$$

Therefore, u is a homomorphism. Now define v as $v : \pi(X, y) \to \pi(Y, x)$ such that $\beta \mapsto \gamma\beta\gamma^{-1}$. Notice that $u \circ v(\alpha) = u(\gamma\alpha\gamma^{-1}) = \gamma^{-1}(\gamma\alpha\gamma^{-1})\gamma \simeq \alpha$. Thus u and v are inverses of each other, and therefore isomorphisms. □

Definition 2.17 (Contractible to a point)**.** A topological space X is contractible to a point if there exists a point x_0 such that $\{x_0\}$ is the deformation retract of X.

Therefore, intuitively, a contractible space can be shrunk to a point. It follows that the fundamental group of a contractible space is trivial.

Definition 2.18. (Simply connected) A topological space X is simply connected if it is path connected and $\pi(X, x) = \{1\}$ for some (hence for all) $x \in X$.

We know that every path in $\mathbb{R}^n$ is homotopic to the constant path via the linear homotopy. Therefore $\mathbb{R}^n$ is simply connected.

2.4. **Effect of a continuous mapping on the Fundamental Group.** Let $\phi : X \to Y$ be continuous. Suppose f_0 and f_1 are homotopic paths in X. Then so are $\phi \circ f_0$ and $\phi \circ f_1$. Let $\phi_*(\alpha)$ be the image of path class α in X. Note that if $f_0, f_1 \in [\alpha]$, then $\phi \circ f_0, \phi \circ f_1 \in \phi_*(\alpha)$. The map ϕ_* has the following properties:

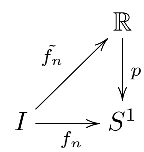
1. $\phi_*(\alpha \cdot \beta) = \phi_*(\alpha) \cdot \phi_*(\beta)$.
2. For any $x \in X, \phi_*(\epsilon_x) = \epsilon_{\phi(x)}$.
3. $\phi_*(\alpha^{-1}) = (\phi_*(\alpha))^{-1}$.
4. If ψ and ϕ are two continuous mappings then, $(\psi\phi)_* = \psi_*\phi_*$.
5. If $\phi : X \to X$ is the identity map then, $\phi_*(\alpha) = \alpha$ for any path α in X i.e., ϕ_* is the identity homomorphism.

So ϕ induces a homomorphism $\phi_* : \pi(X, x) \to \pi(Y, y)$. Notice that if $\phi : X \to Y$ is a homeomorphism, then the induced map $\phi_* : \pi(X, x) \to \pi(Y, y)$ is an isomorphism. Therefore if two topological spaces are homeomorphic, then they have isomorphic fundamental groups. In fact, we have a much stronger result: If two spaces are homotopy equivalent, then their fundamental groups are isomorphic.

2.5. **Fundamental group of a circle.** We will now see how the fundamental group of a circle looks like. What are the closed paths in a circle? We can go around the circle once (from some base point). Call this path α. We can traverse α in the opposite direction to get α^{-1}. Going around the circle twice is the path α^2. Similarly, going around the circle thrice is α^3 and so on. These are all the closed paths in a circle and we can therefore say that the fundamental group of a circle is infinite cyclic. We give a rigorous proof of this statement below.

Theorem 2.19. *The fundamental group of a circle $\pi(S^1, 1) \cong \mathbb{Z}$.*

Proof. Let f_n be a loop around 1 in S^1 defined as follows: $f_n : I \to S^1$ such that $f_n(s) = e^{2\pi ins}$. This map gives us a loop in S^1 which goes around the circle n times (from the base point 1). Notice that $[f_m][f_n] = [f_{m+n}]$. This induces a homomorphism $\phi : \mathbb{Z} \to \pi(S^1, 1)$ given by: $\phi(n) = [f_n]$. We will prove that the map ϕ is an isomorphism.
Now, define $p : \mathbb{R} \to S^1$ by $p(s) = e^{2\pi is}$. This map wraps the interval $[n, n+1]$ around the circle starting at 1 and going around anticlockwise. Let $\tilde{f}_n : I \to \mathbb{R}$ be defined as follows: $\tilde{f}_n(s) = ns$. Notice that $f_n = p \circ \tilde{f}_n$. We therefore have the following commutative diagram:



The idea now is to lift a path in S^1 to a unique path in $\mathbb{R}$ using the fact that S^1 is locally homeomorphic to the real line. So we next claim the following: For any path $f : I \to S^1$ such that $f(0) = 1$, there exists a unique path $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $p \circ f = \tilde{f}$. Let U be a small open neighborhood in S^1. Observe that each path component of $f^{-1}U$ in $\mathbb{R}$ is homeomorphic to U. Cover S^1 with small neighborhoods say $\{U_\alpha\}$. Then $\{f^{-1}(U_\alpha)\}$ is a covering of I. We can now divide the unit interval I into subintervals $[0, t_1], [t_1, t_2], \ldots, [t_{n-1}, 1]$ where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that for all $0 \leq i \leq n-1$, $f[t_i, t_{i+1}]$ lies in U_α for some α. To see this we need a simple concept of the "Lebesgue number." We will not prove its existence but only state it.

Lemma 2.20. *For every open cover μ of a compact metric space X, there is a positive real number λ, called the Lebesgue number such that every subset of diameter less than λ is contained in some subset of μ.*

Let λ be the Lebesgue number of the covering $\{f^{-1}(U_\alpha)\}$ of I. Divide the unit interval into any subintervals of length less that λ. It follows from the definition of the Lebesgue number that f carries each subinterval to one of the open intervals of S^1. Since the above holds, we have a unique lift of f in $\mathbb{R}$ for each subinterval $[t_i, t_{i+1}]$. This lifting of each subinterval is determined by the lifting of its initial point.
Next, define a function $\psi : \pi(S^1, 1) \to \mathbb{Z}$ by $\psi([f]) = \tilde{f}(1)$, the endpoint of the lifted path. Note that $\tilde{f}(1)$ is an integer since $f(1) = p(\tilde{f}(1)) = 1$. To show that the map is well defined, we must show that it is independent of the choice of the path in $[f]$. Suppose H is a homotopy between two maps f and g in $[f]$. We must show that their respective lifting maps $\tilde{f}$ and $\tilde{g}$ are homotopic via some homotopy, say $\tilde{H}$ such that $H = p\tilde{H}$. We use a similar Lebesgue number argument for the interval $I \times I$ to construct $\tilde{H}$. Let $H : I \times I \to S^1$ such that:

$$
\begin{aligned}
H(s, 0) &= f(s) \\
H(s, 1) &= g(s) \\
H(0, t) &= f(0) = g(0) = 1 \\
H(1, t) &= f(1) = g(1) = 1
\end{aligned}
$$

We can divide $I \times I$ into sub-squares such that H maps each small rectangle into some small neighborhood in S^1. Proceed sub-square by sub-square to construct $\tilde{H}$. Note that H maps a small rectangle into a small neighborhood of S^1. Since the path components of $p^{-1}(U)$ are homeomorphic to U, we can construct $\tilde{H}$ uniquely. Note that since the lifted path is unique, $\tilde{H}(s,0) = \tilde{f}(s)$ and $\tilde{H}(s,1) = \tilde{g}(s)$. We know that $\psi([f_n]) = \tilde{f}_n(1) = n$. Therefore the map $\psi \circ \phi : \mathbb{Z} \to \mathbb{Z}$ is the identity map. Thus, ψ is surjective and ϕ is injective. It remains to show that ψ is injective which would imply ϕ is onto. Suppose that $\psi([f]) = \psi([g])$. This implies that $\tilde{f}(1) = \tilde{g}(1)$. Thus, $\tilde{f} \cdot \tilde{g}^{-1}$ is a loop in $\mathbb{R}$ around 0. Recall that the fundamental group of the real line is trivial. So $[\tilde{f} \cdot \tilde{g}^{-1}] = [c_0]$ where c_0 is the constant path at 0 in $\mathbb{R}$. This further implies that $p_*([\tilde{f} \cdot \tilde{g}^{-1}]) = [f][g^{-1}] = [c_1]$. Thus, $[f] = [g]$. We therefore conclude that $\phi : \mathbb{Z} \to \pi(S^1, 1)$ is an isomorphism.

□

We can now answer the first question we asked at the beginning of the paper. We know that $\mathbb{R}^n \backslash (0,0)$ is homotopy equivalent to S^1. Since the fundamental group of $\mathbb{R}^n$ is trivial, while that of S^1 is $\mathbb{Z}$, they are not homotopy equivalent as topological spaces.

3. Recap of some group theoretic concepts

This section will include a quick recap of group actions and propositions related to it. We will state them without the proof.

Definition 3.1. (Group action) Suppose G is a group and X is a non empty set. We say that G is acting on X if there is a map $\phi : G \times X \to X$, where $(g,x) \mapsto g \cdot x$, such that the following hold:

(1) $e \cdot x = x$ for all $x \in X$, where e is the identity element of G.
(2) $(g_1 g_2) \cdot x = g_1(g_2 \cdot x)$

We say that G acts on X (from the left). We will refer to X as the (left) G space. Note that every element of g acts on x to give us a new element $x' \in X$. So, a group action induces a map $G \to \Gamma(X)$, the set of bijections of X.

Definition 3.2 (Isotropy subgroup)**.** Suppose G is acting on X. The stabilizer or the isotropy subgroup of an element $x \in X$ is the set $\{g \in G : g \cdot x = x\}$. We denote this set by G_x.

Note that we have an equivalence relation on X, $x \sim x'$ if and only if there exists an element $g \in G$ such that $x' = g \cdot x$. The equivalence class of $x = \{x' | x' = g \cdot x\}$. This is called the **orbit** of x and we denote it by $G \cdot x$. Thus X can be written as a disjoint union of all the orbits of its elements.

Proposition 3.3. *Suppose G is a group acting on the set X. Then $G \cdot x_i \cong G/G_{x_i}$ for all $x_i \in X$.*

Remark 3.4. We say that a group acts **freely** on X if it acts without any fixed points, i.e., the set G_x is trivial for any $x \in X$.

The action of G is called **transitive** if there is only one orbit of X, i.e., for all $x \in X, G \cdot x = X$. If G acts on X transitively, then we call X a **homogeneous** G **space**.

Proposition 3.5. *A group A of automorphisms of a (left) homogeneous G space is the entire group of automorphisms if and only if for any two points x and $y \in X$ which have the same isotropy subgroup, there exists an automorphism $\phi \in A$ such that $\phi(x) = y$.*

Recall that the normalizer of a subgroup H of G is the set:

$$N(H) = \{g \in G | gHg^{-1} = H\}.$$

We denote the group of all automorphisms of X as $Aut(X)$.

Proposition 3.6. *Let X be a homogeneous G space and H be the isotropy subgroup of some $x \in X$. Then the group $Aut(X) \cong N(H)/H$.*

See [1, p. 257] for the proofs of the above propositions. Keeping the above propositions in mind, we can now move on to the concept of covering spaces.

4. Covering spaces

4.1. **Definition and Examples.**

Definition 4.1. Let X be a topological space. A covering of X is a space $\tilde{X}$ together with a continuous map $p : \tilde{X} \to X$ such that the following holds: Each point $x \in X$ has a path connected, open neighborhood U, such that each path component of $p^{-1}(U)$ is mapped homeomorphically in U by p.

We will refer to U as an elementary neighborhood of any point $x \in U$. Let us see some examples of covering spaces.

Example 4.2. If X is a topological space and $i : X \to X$ is the identity map, then X is trivially a covering of itself.

Example 4.3. Let $X = S^1$. Then we have a map $p : \mathbb{R} \to S^1$ defined by:

$$p(t) = (\cos t, \sin t).$$

This map "wraps" the real line around the circle. So $(\mathbb{R}, p)$ is a covering space of S^1 (see Figure 4).

In the theorems that follow, it will always be useful to have this example in mind to understand things.

Example 4.4. We can have a map from a circle which maps around itself n times. If we have polar coordinates (r, θ), then for the unit circle consider the following map:

$$p_n(1, \theta) = (1, n\theta).$$

This is a covering map for the circle as well.

Example 4.5. A torus can be covered by the plane $\mathbb{R}^2$ or a cylinder. This can be proved using the fact that, if $(\tilde{X}, p)$ is a covering of X and $(\tilde{Y}, q)$ is a covering of the space Y, then $(\tilde{X} \times \tilde{Y}, p \times q)$ is a covering of $X \times Y$, where the map $p \times q$ is defined as: $(p \times q)(x, y) = (px, qy)$. We can therefore construct a covering of a torus. A torus is essentially the space $S^1 \times S^1$. So the plane $\mathbb{R} \times \mathbb{R}$ and the cylinder $(\mathbb{R} \times S^1)$ is a covering of the torus, with the product of the maps defined in the previous examples.

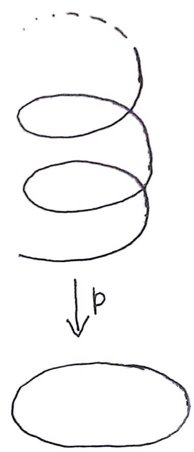


FIGURE 4

4.2. **Lifting of paths to a covering space of a topological space.** We previously saw that a continuous map between two spaces X and Y induces a homomorphism between their fundamental groups. Given any two homotopic paths in X, their continuous images are homotopic in Y. Now we ask whether the converse is true. Given any two homotopic paths in Y, are their pre-images homotopic? It turns out this is true for covering maps.

Lemma 4.6. *Let $(\tilde{X}, p)$ be a covering of the space X, let $x_0 \in x$ and $\tilde{x_0} \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$. Then for any path $f : I \to X$ in X with initial point x_0, there exists a unique path $g : I \to \tilde{X}$ in $\tilde{X}$ with initial point $\tilde{x}_0$ such that $pg = f$.*

Proof. Let U be any elementary neighborhood and $f \subset U$. Let V be the path component of $p^{-1}(U)$ which contains $\tilde{x}_0$. Since p maps V homeomorphically into U, there exists a unique path g in V with the initial point as x_0 such that $pg = f$. Now assume f is not wholly contained in U. In this case, the idea is to express f as a product of shorter paths each of which is contained in an elementary neighborhood. We can then apply the above argument. Let $\{U_i\}$ be a covering of X by elementary neighborhoods. Then, $\{f^{-1}(U_i)\}$ is an open covering of I. Let λ be the Lebesgue number of the covering. Now, choose n such that $\frac{1}{n} < \lambda$. Divide the interval I into the closed subintervals $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]$. Since, the diameter of these intervals is less than λ, f maps each of these intervals inside U_i. We can now define g over these subintervals successively using the argument we did above. □

The uniqueness of the path comes from the following lemma:

Lemma 4.7. *Let $(\tilde{X}, p)$ be a covering of X. Given any two maps $f_0, f_1 : I \to \tilde{X}$ such that $pf_0 = pf_1$, the set $\{y \in I | f_0(y) = f_1(y)\}$ is either empty or all of I.*

Proof. Recall that in a connected space, the set which is both open and closed is either the empty set ϕ or the whole space. We will prove that the set $A = \{y \in I | f_0(y) = f_1(y)\}$ is both open and closed. First we will see that it is closed. Let y be a point of the closure of this set and let $x = pf_0(y) = pf_1(y)$. Assume $f_0(y) \neq f_1(y)$. We will see that this leads to a contradiction. Let U be an elementary neighborhood of x and V_0 and V_1 be the components of $p^{-1}(U)$ which contain $f_0(y)$ and $f_1(y)$ respectively. Since f_0 and f_1 are both continuous, we can find a neighborhood W of y such that $f_0(W) \subset V_0$ and $f_1(W) \subset V_1$. Note that $V_0 \cap V_1 = \phi$. This is a contradiction to the fact that every neighborhood of y must intersect the set A. Thus A contains all its closure points and is therefore closed. Analogously we can argue that every point in A is an interior point and therefore the set is open. Since f_0 and f_1 agree on at least one point in I, i.e., $f_0(0) = f_1(0) = \tilde{x}_0$, they have to be the same everywhere. □

Lemma 4.8. *Let $(\tilde{X}, p)$ be a covering space of X. Let $g_0, g_1 : I \to \tilde{X}$ be paths in $\tilde{X}$ which have the same initial point. If $pg_0 \simeq pg_1$, then $g_0 \simeq g_1$. In particular g_0 and g_1 have the same terminal point.*

Proof. Let $\tilde{x}_0$ be the initial point of g_1 and g_1. Since, $pg_0 \sim pg_1$, there exists a map $F : I \times I \to X$ such that:

$$\begin{aligned} F(s,0) &= pg_0(s) \\ F(s,1) &= pg_1(s) \\ F(0,t) &= pg_0(0) = p(\tilde{x}_0) \\ F(1,t) &= pg_0(1) \end{aligned}$$

By using a Lebesgue number argument again, we can find the partition $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that F maps each small rectangle $[s_{i-1} \times s_i] \times [t_{j-1} \times t_j]$ into some elementary neighborhood in X. We have to prove that $g_0 \sim g_1$. We will prove this by showing that there exists a unique map $G : I \times I \to \tilde{X}$ such that $pG = F$ and $G(0,0) = \tilde{x}_0$. First we define G over the small rectangle $[s_1, 0] \times [0, t_1]$. Note that F maps this small rectangle to an elementary neighborhood of $p(\tilde{x}_0)$ say U. Since components of $p^{-1}(U)$ are homeomorphically mapped by p into U, we can construct G as in Lemma 4.6. Now, we can extend this successively to other rectangles. Uniqueness of G comes from Lemma 4.7. Note that $G(s,0) = g_0(s)$ and $G(s,1) = g_1(s)$. This follows from the uniqueness assertion of Lemma 4.6. Also, $G(0,0) = g_0(0) = g_1(0) = \tilde{x}_0$. □

The consequence of these lemmas is a very important result:

Theorem 4.9. *Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0)$. Then the induced homomorphism $p_* : \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is injective.*

Proof. Let $p : \tilde{X} \to X$ be the covering map, then, $p_* : \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is the induced homomorphism. Let $[\alpha]$ and $[\beta]$ be two path classes in $\tilde{X}$. Suppose g_α and g_β are paths in $[\alpha]$ and $[\beta]$ respectively. Let $p_*[\alpha] = p_*[\beta]$. This implies that $pg_\alpha \simeq pg_\beta$. It follows from Lemma 4.8 that $g_\alpha \simeq g_\beta$ in $\tilde{X}$. So, $[\alpha] = [\beta]$. Thus the map is injective. □

Now, suppose $\tilde{x}_0$ and $\tilde{x}_1$ are points in $\tilde{X}$ such that $p(\tilde{x}_0) = p(\tilde{x}_1) = x_0$. How are the images of $p_* : \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ and $p_* : \pi(\tilde{X}, \tilde{x}_1) \to \pi(X, x_0)$ related?
To answer this, we choose a class γ of paths in $\tilde{X}$ starting from $\tilde{x}_0$ and ending at $\tilde{x}_1$. This defines an isomorphism $\phi : \pi(\tilde{X}, \tilde{x}_0) \to \pi(\tilde{X}, \tilde{x}_1)$ given by : $\phi(\alpha) = \gamma^{-1}\alpha\gamma$.

We thus have the following commutative diagram:

$$\begin{array}{ccc} \pi(\tilde{X},\tilde{x}_0) & \xrightarrow{p_*} & \pi(X,x_0) \\ \phi\downarrow & & \downarrow\psi \\ \pi(\tilde{X},\tilde{x}_1) & \xrightarrow[p_*]{} & \pi(X,x_0) \end{array}$$

Here, $\psi(\beta) = (p_*\gamma)^{-1}\beta(p_*\gamma)$. Notice that $p(\tilde{x}_0) = p(\tilde{x}_1) = x_0$, which is the initial and the terminal point of $p_*\gamma$. Therefore $p_*\gamma$ is a closed path and belongs to $\pi(X,x_0)$. So we conclude that images of $\pi(\tilde{X},\tilde{x}_0)$ and $\pi(\tilde{X},\tilde{x}_1)$ are conjugate subgroups of $\pi(X,x_0)$.

Next question we ask is the following: Can every subgroup in a conjugacy class of $p_*\pi(\tilde{X},\tilde{x}_0)$ be obtained as the image of $p_*\pi(\tilde{X},\tilde{x}_1)$ for some choice of $\tilde{x}_1$? It turns out that the answer is yes! We have the following theorem:

Theorem 4.10. *Let $(\tilde{X},p)$ be a covering space of X and let $x_0 \in X$. Then the subgroups $p_*\pi(\tilde{X},\tilde{x})$ for $\tilde{x} \in p^{-1}(x_0)$ are exactly a conjugacy class of subgroups of $\pi(X,x_0)$.*

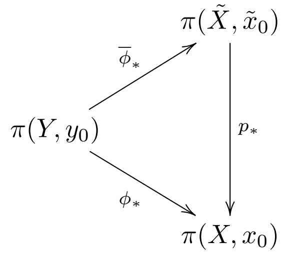
Proof. Any subgroup in the conjugacy class of $p_*(\tilde{X},\tilde{x}_0)$ looks like $\alpha^{-1}[p_*(\tilde{X},\tilde{x}_0)]\alpha$, for some $\alpha \in \pi(X,x_0)$. Choose a closed path $f : I \to X$ representing α. By Lemma 4.7, we have a path $g : I \to \tilde{X}$ covering f with initial point $\tilde{x_0}$, such that $p_*g = f$. Let $\tilde{x}_1$ be the terminal point of the lifted path. Then $p_*\pi(\tilde{X},\tilde{x}_1) = \alpha^{-1}[p_*(\tilde{X},\tilde{x}_0)]\alpha$. □

4.3. **Lifting of arbitrary maps to a covering space.** Previously, we had a path from the unit interval I in X and we lifted it to a path in the covering space $\tilde{X}$. We now generalize this and study the lifting of paths of X from an arbitrary connected space Y. We introduce some notation first. If X and Y are topological spaces such that $x \in X$ and $y \in Y$, then, $f : (X,x) \to (Y,y)$ means that f is a continuous map from X to Y such that $f(x) = y$.

Let $(\tilde{X},p)$ be a covering space of X and $\tilde{x}_0 \in \tilde{X}$ such that $x_0 = p(\tilde{x}_0)$. Let $y_0 \in Y$ and $\phi : (Y,y_0) \to (X,x_0)$ be a continuous map. We want to find the condition under which there exists a map $\overline{\phi} : (Y,y_0) \to (\tilde{X},\tilde{x}_0)$ such that the following diagram is commutative:

$$\begin{array}{ccc} & & (\tilde{X},\tilde{x}_0) \\ & \overset{\overline{\phi}}{\nearrow} & \downarrow p \\ (Y,y_0) & \xrightarrow[\phi]{} & (X,x_0) \end{array}$$

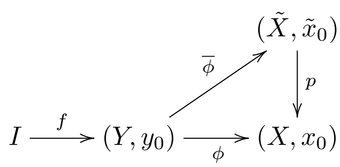
If $\overline{\phi}$ exists, then we say ϕ can be lifted to $\tilde{X}$. We call $\overline{\phi}$ to be a lifting of ϕ. Note that if $\overline{\phi}$ exists then we have the following commutative diagram of group homomorphisms:



Since p_* is $1-1$, for the diagram to be commutative we need $Im(\phi_*) \subset Im(p_*)$. This condition is also sufficient and we have the following theorem:

Theorem 4.11. *Let $(\tilde{X}, p)$ be a covering of X, Y be a connected and locally path connected space, and $y_0 \in Y, \tilde{x_0} \in \tilde{X}$ such that $x_0 = p(\tilde{x}_0)$. Given a map $\phi : (Y, y_0) \to (X, x_0)$, there exists a lifting $\overline{\phi} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if $\phi_* \pi(Y, y_0) \subset p_* \pi(\tilde{X}, \tilde{x}_0)$.*

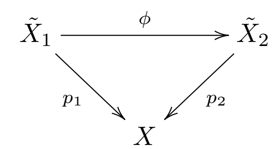
Proof. We argued above that condition is necessary. Now we will show that the condition is sufficient for the diagram to be commutative. In order to do so, we will define $\overline{\phi}$. Assume $\overline{\phi}$ exists. Let y be any point in Y. Since Y is path connected, choose a path $f : I \to Y$ with initial point y_0 and terminal point y. Consider the paths ϕf and $\overline{\phi} f$ in X and $\tilde{X}$ respectively. $\overline{\phi} f$ is a lifting of ϕf and $\overline{\phi}(y)$ is the terminal point of the path ϕf. Define $\overline{\phi} : (Y, y_0) \to (\tilde{X}, \tilde{x})$ as follows: Given any point $y \in Y$, choose a path $f : I \to Y$ with the initial point y_0 and the terminal point y. Then, ϕf is a path in X. By Lemma 4.6, there exists a path $g : I \to \tilde{X}$ such that the initial point is $\tilde{x}_0$ and $pg = \phi f$. Now, define $\overline{\phi}(y)$ to be the terminal point of g. We will now justify this definition by showing that $\overline{\phi}(y)$ is independent of the choice of the path f. By Lemma 4.8, we can change f by an equivalent path without changing the definition of $\overline{\phi}(y)$. So $\overline{\phi}(y)$ only depends on the equivalent classes of paths in Y. Suppose α and β are two different equivalent classes of paths in Y from y_0 to y. Then, $\alpha\beta^{-1}$ is a closed path based at y_0; hence, $\alpha\beta^{-1} \in \pi(Y, y_0)$. This implies, $\phi_*(\alpha\beta^{-1}) \in p_*\pi(\tilde{X}, \tilde{x})$ (using the hypothesis of the theorem). This further implies that if $(\phi_*\alpha)(\phi_*\beta)^{-1}$ is lifted to a path in $\tilde{X}$ starting at $\tilde{x}_0$, the result is a closed path in $\tilde{X}$. Thus if $\phi_*\alpha$ and $\phi_*\beta$ are each lifted to paths in $\tilde{X}$ starting at $\tilde{x}_0$, they will have the same terminal point.
It is now left to prove that $\overline{\phi}$ is continuous. Let $y \in Y$ and U be an arbitrary neighborhood of $\overline{\phi}(y)$. We need to show that there exists a neighborhood V of y such that $\overline{\phi}(V) \subset U$. Choose an arbitrary neighborhood U' of $p\overline{\phi}(y) = \phi(y)$ such that $U' \subset p(U)$. Let W be the path component of $p^{-1}(U')$ which contains $\overline{\phi}(y)$. Now let U'' be an elementary neighborhood of $\phi(y)$ such that $U'' \subset p(U \cap W)$. Notice that the path component of $p^{-1}(U'')$ which contains $\overline{\phi}(y)$ is a subset of U. Since ϕ is continuous we can choose V such that $\phi(V) \subset U''$, so that $\overline{\phi}(V) \subset U$.



□

4.4. **Homomorphism between covering spaces.** How are two covering spaces of a topological space related? Can we define a map between two covering spaces to identify them up to isomorphism?

Definition 4.12. Let $(\tilde{X}, p_1)$ and $(\tilde{X}, p_2)$ be covering spaces of X. A homomorphism of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ is a continuous map $\phi : \tilde{X}_1 \to \tilde{X}_2$ such that the following diagram is commutative:



Definition 4.13. A homomorphism ϕ of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ is called an isomorphism if there exists a homomorphism ψ of $(\tilde{X}_2, p_2)$ into $(\tilde{X}_1, p_1)$ such that $\phi\psi$ and $\psi\phi$ are identity maps.

An automorphism is an isomorphism of a covering space to itself. Set of all automorphisms of $\tilde{X}$ form a group under composition.

Lemma 4.14. *Let ϕ_0 and ϕ_1 be homomorphisms of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$. If there exists any point $x \in \tilde{X}_1$ such that $\phi_0(x) = \phi_1(x)$, then $\phi_0 = \phi_1$.*

Proof. This follows from Lemma 4.7 □

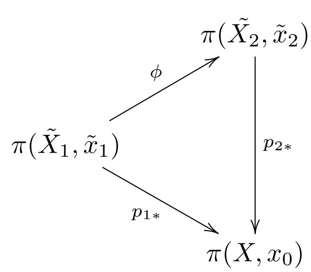
So if ϕ has a fixed point then it follows from the above lemma that ϕ is the identity map.

Corollary 4.15. *If $\phi \in A(\tilde{X}, p)$ and $\phi \neq \mathbb{I}$, then ϕ has no fixed points.*

Next we will see a special case of Theorem 4.11.

Lemma 4.16. *Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X and let $\tilde{x}_i \in \tilde{X}_i$ for $i = 1, 2$. such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$. Then there exists a homomorphism ϕ of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ such that $\phi(\tilde{x}_1) = \tilde{x}_2$ if and only if $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) \subset p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$.*

Proof. The proof follows by applying Theorem 4.11 to the following diagram.



□

The following two corollaries now are a direct consequence of Lemma 4.16.

Corollary 4.17. *Under the conditions of the above lemma, there exists an isomorphism ϕ of $(\tilde{X}_1, p_1)$ onto $(\tilde{X}_2, p_2)$ such that $\phi(\tilde{x_1}) = \phi(\tilde{x_2})$ if and only if $p_{1*}\pi(\tilde{X}_1, \tilde{x_1}) = p_{2*}\pi(\tilde{X}_2, \tilde{x_2})$.*

Corollary 4.18. *Let $(\tilde{X}, p)$ be a covering space of X and $\tilde{x_1}, \tilde{x_2} \in p^{-1}(x_0)$ where $x_0 \in X$. There exists an automorphism $\phi \in A(\tilde{X}, p)$ such that $\phi(\tilde{x}_1) = \tilde{x}_2$ if and only if $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$.*

Theorem 4.19. *Two covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ of X are isomorphic if and only if for any two points $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x_2} \in \tilde{X}_2$ such that $p_1(\tilde{x_1}) = p_2(\tilde{x_2}) = x_0$, the subgroups $p_{1*}\pi(\tilde{X}_1, \tilde{x_1})$ and $p_{2*}\pi(\tilde{X}_2, \tilde{x_2})$ belong to the same conjugacy class.*

Proof. From Corollary 4.17, we know that two covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are isomorphic if and only if $p_*\pi(\tilde{X}, \tilde{x_1}) = p_*\pi(\tilde{X}, \tilde{x_2})$. By Theorem 4.10, they belong to the same conjugacy class of subgroup of $\pi(X, x_0)$. ☐

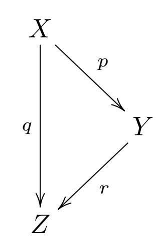
Therefore, conjugacy class of subgroups completely determine the covering spaces up to isomorphism. Now, what does it mean to have a homomorphism between two covering spaces? Does one of them act as a covering space for the other as well?

Lemma 4.20. *Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X and ϕ be a homomorphism of the first covering space into the second. Then $(\tilde{X}_1, \phi)$ is a covering of $\tilde{X}_2$.*

Proof. Observe that any point $x \in X$ has an open neighborhood U which is the elementary neighborhood of both the covering spaces. We can obtain such a neighborhood of x by choosing elementary neighborhoods U_1 and U_2 of x for the coverings $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ respectively, and then take $U = U_1 \cap U_2$. Next we prove that ϕ maps $\tilde{X}_1$ onto $\tilde{X}_2$. Let y be any point of $\tilde{X}_2$. We need to show that there exists a point x of $\tilde{X}_1$ such that $\phi(x) = y$. Choose a base point $x_1 \in \tilde{X}_1$ and let $x_2 = \phi(x_1)$, $x_0 = p_1(x_1) = p_2(x_2)$. Choose a path f in $\tilde{X}_2$ with initial point x_2 and terminal point y. Let $g = p_2 f$ be its image in X. By Lemma 4.6, there exists a unique path h in $\tilde{X}_1$ with initial point x_1 such that $p_1 h = g$. Let x be the terminal point of h. Notice that ϕh and f both have the same initial point and $p_2\phi h = g = p_2 f$. So by the uniqueness assertion in Lemma 4.6, $\phi h = f$. Thus, $\phi(x) = y$. ☐

Let $(\tilde{X}, p)$ be a covering space which is simply connected. If $(\tilde{X}', p')$ is any other covering space of X, then there exists a homomorphism ϕ of $(\tilde{X}, p) \hookrightarrow (\tilde{X}', p')$ since $p_*\pi(\tilde{X}, p) = \{1\} \subset p_*\pi(\tilde{X}', p')$. Thus $(\tilde{X}, p)$ is a covering for all the other covering spaces. We will call it the *universal cover*.
If we have two universal coverings of a space, say (X_1, p_1) and (X_2, p_2) then their fundamental groups are trivial. Thus, $p_*\pi(X_1, p_1) = p_*\pi(X_2, p_2) = p_*\{1\}$. By Corollary 4.17, they are isomorphic.

Proposition 4.21. *Consider the following commutative diagram of spaces and continuous maps. Assume (X, p) is a covering of Y and (X, q) is a covering of Z. Then (Y, r) is a covering of Z.*



Proof. Let $z \in Z$. Since the diagram is commutative, we have that $q^{-1}(z) = p^{-1}r^{-1}(z)$. Now, Let U be an elementary neighborhood of z and V be an elementary neighborhood of $r^{-1}(z)$. We know that the intersection $q^{-1}(U)$ and $p^{-1}r^{-1}(V)$ is non empty. Note that each path component of this intersection maps homeomorphically to an open set in U and V (say U_1 and V_1 respectively). This gives us a homeomorphism between U_1 and V_1. Thus U_1 acts as an elementary neighborhood for the space Y and thus, Y is a covering space of Z. □

4.5. **Action of the group $\pi(X, x)$ on the set $p^{-1}(x)$ for any $x \in X$.**

Definition 4.22. Let $(\tilde{X}, p)$ be a covering space of X. For any point $\tilde{x} \in p^{-1}(x)$ and any $\alpha \in \pi(X, x)$, define $\tilde{x} \cdot \alpha \in p^{-1}(x)$ as follows: Let $\tilde{\alpha}$ be the lifting of α in $\tilde{X}$ with the initial point $\tilde{x}$ such that $p_*(\tilde{\alpha}) = \alpha$. Define $\tilde{x} \cdot \alpha$ to be the terminal point of the path class $\tilde{\alpha}$. We can see in Figure 5 that the closed path α which goes around the circle once, acts on $\tilde{x}_1$ such that $\alpha \cdot \tilde{x}_1 = \tilde{x}_2$.

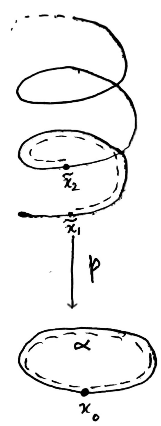


Figure 5

It follows from the definition that:

(1) $\tilde{x} \cdot 1 = \tilde{x}$
(2) $(\tilde{x} \cdot \alpha) \cdot \beta = \tilde{x} \cdot (\alpha\beta)$

Therefore, this defines a right group action of $\pi(X, x)$ on the set $p^{-1}(x)$. Now, let $\tilde{x}_0$ and $\tilde{x_1} \in p^{-1}(x)$. Since $\tilde{X}$ is path connected, there exists a path class $\tilde{\alpha}$ in $\tilde{X}$ with the initial point $\tilde{x}_0$ and terminal point $\tilde{x}_1$. Let $\alpha = p_*(\tilde{\alpha})$. Note that α is the equivalence class of closed paths and $\tilde{x}_0 \cdot \alpha = \tilde{x}_1$. We just proved that the group $\pi(X, x)$ acts transitively on the set $p^{-1}(x)$.

Proposition 4.23. *The isotropy subgroup corresponding to any $\tilde{x} \in p^{-1}(x)$ is the subgroup $p_*\pi(\tilde{X}, \tilde{x})$ of $\pi(X, x)$.*

Proof. Note that the isotropy subgroup of any $\tilde{x} = \{\alpha | \alpha \cdot \tilde{x} = \tilde{x}, \alpha \in \pi(X, x)\}$. This is exactly $p_*\pi(X, \tilde{x})$. □

Proposition 4.24. *As a right $\pi(X, x)$-space, $p^{-1}(x) \cong \pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$.*

Proof. This follows from Proposition 3.3. □

4.6. **Action of $Aut(\tilde{X}, p)$ on $p^{-1}(x)$ as a right $\pi(X, x)$ space.**

Proposition 4.25. *For any automorphism $\phi \in A(\tilde{X}, p)$, any point $\tilde{x} \in p^{-1}(x)$, and any $\alpha \in \pi(X, x)$, $\phi(\tilde{x} \cdot \alpha) = \phi(\tilde{x}) \cdot \alpha$.*

The above proposition tells us that the automorphism group acts on $p^{-1}(x)$ as a right $\pi(X, x)$-space i.e., the action of the automorphism group commutes with the action of $\pi(X, x)$.

Proof. Lift α to a path $\tilde{\alpha}$ in $\tilde{X}$ with initial point $\tilde{x}$, such that $p_*(\tilde{\alpha}) = \alpha$. Note that $\tilde{x} \cdot \alpha$ is the terminal point of α. Now consider the path $\phi_*(\alpha)$ in $\tilde{X}$. Its initial point is $\phi(\tilde{x})$ and the terminal point is $\phi(\tilde{x} \cdot \alpha)$. Observe that:

$$p_*(\phi_*(\tilde{\alpha})) = (p\phi)_*(\tilde{\alpha}) = p_*(\tilde{\alpha}) = \alpha.$$

This implies that $\phi_*(\tilde{\alpha})$ is also a lifting of α. By definition, $(\phi\tilde{x}) \cdot \alpha$ is the terminal point $\phi_*(\tilde{\alpha})$. Therefore, $(\phi\tilde{x}) \cdot \alpha = \phi(\tilde{x} \cdot \alpha)$. □

Theorem 4.26. *Let $(\tilde{X}, p)$ be a covering space of X. Then the group of automorphisms, $A(\tilde{X}, p)$ is naturally isomorphic to the group of automorphisms of the set $p^{-1}(x)$ considered as a right $\pi(X, x)$ space.*

Proof. Note that if ϕ is an automorphism of $\tilde{X}$, then $\phi|_{p^{-1}(x)}$ is an automorphism of $p^{-1}(x)$. First we will prove that the map : $\phi \mapsto \phi|_{p^{-1}}(x)$ is one to one. Suppose $\phi|_{p^{-1}(x)} = \psi|_{p^{-1}(x)}$. This implies, $\phi\psi^{-1}|_{p^{-1}(x)} = \mathbb{I}$. Since ϕ and ψ do not have any fixed points, it follows that $\phi\psi^{-1} = \mathbb{I}$. Thus $\phi = \psi$. We next need to prove that the map is onto. Suppose ϕ is an automorphism of $p^{-1}(x)$ such that $\phi \cdot \tilde{x_1} = \tilde{x_2}$ where $\tilde{x_1}, \tilde{x_2} \in p^{-1}(x)$. Then $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$. From Corollary 4.18, it follows that there exists an automorphism $\psi \in Aut(\tilde{X}, p)$ such that $\psi(\tilde{x}_1) = \tilde{x}_2$. We conclude that the group $Aut(\tilde{X}, p)$ of automorphisms is an entire group of automorphisms by Proposition 3.5. Therefore, the map is onto. □

Corollary 4.27. *For any point $x \in X$ and any $\tilde{x} \in p^{-1}(x)$, the automorphism group $A(\tilde{X}, p)$ is isomorphic to the quotient group $N(p_*\pi(\tilde{X}, \tilde{x}))/p_*\pi(\tilde{X}, \tilde{x})$.*

Proof. This is a direct consequence of Proposition 3.6 and Theorem 4.26. □

Definition 4.28 (Regular covering space)**.** A class of covering spaces for which $p_*\pi(\tilde{X}, x)$ is a normal subgroup of $\pi(X, x)$ is called a regular covering space.

Corollary 4.29. *If $(\tilde{X}, p)$ is a regular covering space of X, then $A(\tilde{X}, p) \cong \pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$.*

Proof. Since $p_*\pi(\tilde{X}, \tilde{x})$ is normal, its normalizer is the whole group $\pi(X, x)$. The above corollary now follows from Corollary 4.27. □

Notice that for a universal covering space, $p_*(\tilde{X}, x) = \{1\}$. Thus $A(\tilde{X}, p) \cong \pi(X, p)$.

4.7. **Regular covering spaces and quotient spaces.** Let $(\tilde{X}, p)$ be a covering space of X. Since p is an open map, X has quotient topology induced by p. We identify the points in $p^{-1}(x)$ to a single base point such that $\tilde{X}/A(\tilde{X}, p) = X$. However $Aut(\tilde{X}, p)$ has to act transitively on $p^{-1}(x)$ in order for X to be a quotient space of $\tilde{X}$. This is not always true.

Lemma 4.30. *The automorphism group $A(\tilde{X}, p)$ operates transitively on $p^{-1}(x)$ if and only if $(\tilde{X}, p)$ is a regular covering of X.*

Proof. Recall that the subgroups $p_*\pi(\tilde{X}, \tilde{x})$ for $\tilde{x} \in p^{-1}(x_0)$ are exactly a conjugacy class of $\pi(X, x_0)$. Since $\tilde{X}$ is a regular covering, if $\tilde{x}_1$ and $\tilde{x}_2 \in p^{-1}(x)$, then $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$. From Corollary 4.18, there exists an $\phi \in Aut(\tilde{X}, p)$ such that $\phi(\tilde{x}_1) = \tilde{x}_2$. This proves that the action is transitive. □

Now, the next question we answer is, when does a group action on a topological space X give rise to a covering map? In order to answer this, we introduce what are called properly discontinuous group actions.

Definition 4.31. A group of homomorphisms G of $\tilde{X}$ is said to be properly discontinuous if every point $\tilde{x} \in \tilde{X}$ has a neighborhood U such that the sets $\phi(U)$ are pairwise disjoint for all $\phi \in G$.

Proposition 4.32. *Let Y be connected, locally path wise connected, topological space and let G be a properly discontinuous group of homomorphisms of Y. Let $p : Y \to Y/G$ denote the natural projection of Y onto its quotient space. Then, (Y, p) is a regular covering space of Y/G, and $G = A(Y, p)$.*

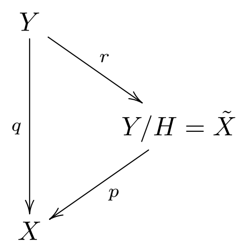
Proof. Let $x \in Y/G$. We must show that x has an elementary neighborhood. Choose a $y \in Y$ such that $p(y) = x$. Since G is properly discontinuous, there exists a neighborhood N of y such that the sets $\phi(N), \phi \in G$, are pairwise disjoint. Since Y is locally path connected, there exists an open neighborhood V of y such that $V \subset N$. Let $U = p(V)$. We claim that U is an elementary neighborhood of x. Note that since p is a open map, U is open and path connected. Also, p is an injection from V onto U. This is because, if $u, v \in V$ such that $p(u) = p(v)$, then the orbit of u is same as the orbit of v. So $\phi(u) = (v)$ for some $\phi \in G$. Since $u, \phi(u) \in V$ and ϕ acts discontinuously, ϕ is the identity homomorphism. Therefore, $u = v$. Thus the map p from V to U is a homeomorphism. If W is any component of $p^{-1}(U)$ different from V, then there exists $\phi \in G$ such that $\phi(V) = W$. Since, ϕ is a homeomorphism of V onto W and $p = p\phi$, it follows that W maps homeomorphically into U. Thus U is an elementary neighborhood and Y is a covering of Y/G.
Now, notice that the G action on Y gives us an automorphism of (Y, p). So, $G \subset Aut(Y, p)$. Suppose x_1 and x_2 are two points in Y. Let ϕ be an element of G such that $\phi(x_1) = x_2$. We can choose an automorphism $\psi \in Aut(Y, p)$ such that $\psi(x_1) = x_2$. It follows that $\phi = \psi$. Thus $Aut(Y, p) \subset G$ and thus $G = Aut(Y, p)$ By construction, G acts transitively on Y. Therefore by Lemma 4.30, (Y, p) is regular. □

Let us see an example to illustrate the above proposition. Let $Y = \mathbb{R}$ and for each integer n define $\phi_n : \mathbb{R} \to \mathbb{R}$ by : $\phi_n(x) = x + n$. Let $G = \{\phi_n | n \in \mathbb{Z}\}$. G is a properly discontinuous group of homomorphisms of $\mathbb{R}$, since for any $x \in R$, if we let $U = (x - \frac{1}{2}, x + \frac{1}{2})$, the neighborhoods $\phi_n(U)$ are pairwise disjoint. Hence by the above proposition $\mathbb{R}$ is a regular covering space of $\mathbb{R}/G$. Recall that $\mathbb{R}/G \simeq S^1$. Once again we proved that the universal covering of a circle is $\mathbb{R}$.

4.8. **One to one correspondence between the subgroups of $\pi(X, x)$ and the covering spaces of X.** We have previously proved that a covering space $(\tilde{X}, p)$ is determined up to isomorphism by the conjugacy class of the subgroup $p_*\pi(\tilde{X}, \tilde{x})$ of $\pi(X, x)$. Now we answer the inverse question: Suppose X is a topological space and we are given a conjugacy class of subgroups of $\pi(X,x)$, does there exist a covering $(\tilde{X},p)$ of X such that $p_*\pi(\tilde{X},\tilde{x})$ belongs to that conjugacy class?

Lemma 4.33. *Let X be a topological space which has a universal covering space. Then for any conjugacy class of subgroups of $\pi(X,x)$, there exists a covering of space $(\tilde{X},p)$ of X such that $p_*\pi(\tilde{X},\tilde{x})$ belongs to the given conjugacy class.*

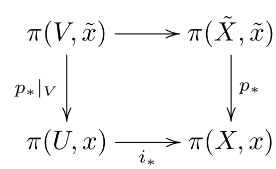
Proof. Let (Y,q) be a universal covering space of X. We know that $\pi(X,x)$ operates transitively and freely on the set $q^{-1}(x)$. We also have that $A(Y,q) \cong \pi(X)$ and $Aut(Y,q)$ operates transitively on the left of the set $q^{-1}(x)$. Choose a subgroup G of $\pi(X,x)$ which belongs to the given conjugacy class. Let H be the subgroup of $A(Y,q)$ defined as follows: $\phi \in H$ if and only if there exists an element $\alpha \in G$ such that $\phi(y) = y \cdot \alpha$ for some $y \in q^{-1}(x)$. Note that $G \cong H$ under the correspondence $\phi \mapsto \alpha$ if and only if $\phi(y) = y \cdot \alpha$. Since H is a subgroup of $A(Y,q)$, it is a properly discontinuous group of homomorphisms of Y. Let $\tilde{X} = Y/H$, and $r : Y \to \tilde{X}$ be the natural projection and $p : \tilde{X} \to X$ induced by q. We have the following commutative diagram:



Notice that (Y,q) is a covering space of Y/H by Proposition 4.32. It follows from Proposition 4.21 that $(\tilde{X},p)$ is a covering space of X. Thus the group $\pi(X,x)$ operates transitively on the right of the set $p^{-1}(x)$. Let $\tilde{x} = r(y) \in p^{-1}(x)$. Since Y is the universal cover, $\pi(Y/H) \cong Aut(Y,r)$, which is equal to H by Proposition 4.32. This gives a map between $Aut(Y,r) \cong H$ in $Aut(Y,q)$ to G in $\pi(X)$. So, p_* maps $\pi(Y/H)$ to G in $\pi(X)$. $\square$

In the above theorem we assumed that the space has a universal covering. However, what if X has no universal covering? We now introduce what is called "semi-locally simply connectedness". We will see that this property is necessary and sufficient for the existence of a universal covering.

Let $(\tilde{X},p)$ be a universal covering space of X. Suppose $x \in X$ and $\tilde{x} \in p^{-1}(x)$. Let U be an elementary neighborhood of x and V be the component of $p^{-1}(U)$ which contains the point $\tilde{x}$. Notice that we have the following commutative diagram involving the fundamental groups:



Since $p|_V$ is a homeomorphism of V onto U, $p_*|_V$ is an isomorphism. We also have that $\pi(\tilde{X},\tilde{x}) = \{1\}$. From the commutativity of the diagram, we can say that i_* is trivial. Therefore, the space X has the following property: Every point $x \in X$ has a neighborhood U such that the homomorphism $\pi(U,x) \to \pi(X,x)$ is trivial.

Definition 4.34. Let X be a topological space, X is called semi-locally simply connected if every $x \in X$ has a neighborhood U such that the homomorphism $\pi(U, x) \to \pi(X, x)$ is trivial.

The argument above tells us the necessity of the property for the existence of the universal cover. We will now show that this property is also sufficient.

Theorem 4.35. *Let X be a topological space which is connected, locally path-wise connected and semi locally simply connected. Then given any conjugacy class of subgroups of $\pi(X, x)$, there exists a covering space $(\tilde{X}, p)$ of X corresponding to the given conjugacy class.*

It suffices to prove here that X has a universal covering. We will only give an idea of the proof on how an early topologist might have constructed the universal covering. Let us make some observations first. Assume for now, that X has a universal covering space $(\tilde{X}, p)$. Choose a base point $\tilde{x}_0 \in \tilde{X}$ and let $x_0 = p(\tilde{x}_0)$. Given any point $y \in \tilde{X}$, there exists a unique path class α with the initial point $\tilde{x}_0$ and the terminal point y. Thus we can associate a point y to the path class $p_*(\alpha)$ in X. We use this observation to give an idea for the construction of a universal covering. Choose a base point $x_0 \in X$, and let $\tilde{X}$ be the set of all path classes which have x_0 as their initial point. We define a map $p : \tilde{X} \to X$ such that $p(\alpha)$ is the terminal point of α in X. We can now define some topology on $\tilde{X}$ so that it is simply connected and it becomes a covering of X. One can see the rigorous proof in [1].
Let us see an example to see how a covering space corresponds to a subgroup of the fundamental group. We know that the real line is the universal covering of a circle. Consider Figure 6. We have a triple cover of a $\tilde{X}$ of a circle. Let α be the path which goes around the circle once (from a base point say x_0). The cover corresponds to going around the circle thrice i.e it is the path α^3. So, $p_*(\pi(\tilde{X})) =< \alpha^3 >$. Note that $< \alpha^3 >$ is exactly $3\mathbb{Z} \subset \mathbb{Z}$. Here we can therefore say that the covering $\tilde{X}$ corresponds to the subgroup $3\mathbb{Z}$ of $\mathbb{Z}$.

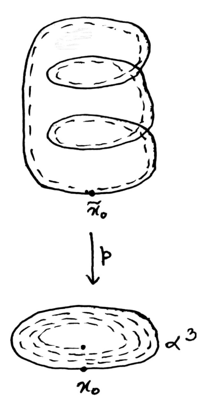


FIGURE 6

5. Conclusions and a way forward

5.1. **The Galois correspondence.** The above theorem proves the existence of the universal covering. The theory we build on covering spaces gives us a one-one correspondence between the covering spaces of a space and the conjugacy classes of its fundamental group. We discuss the idea of the Galois correspondence to draw the analogy between the two theories. We follow the discussion as given in [3]. Suppose K is a field and $L : K$ is a Galois extension. To every intermediate field M, we associate $Aut(L : M)$, the group of all $M-$ automorphisms of L. So, $K \mapsto Aut(K : L)$, the whole Galois group, while $L \mapsto Aut(L : L)$, the identity automorphism of L. Conversely, to each subgroup H of the Galois group, we associate the field $\{x \in L | \alpha(x) = x \text{ for all } \alpha \in H\}$. Therefore, the Galois correspondence gives us a one - one correspondence between the subgroups of the Galois group and the intermediate field. In particular, if M and N are two intermediate subfields such that $M \subset N$, then $Aut(N : L) \subset Aut(M : L)$. The inclusion is also reversed in the case of covering spaces. The trivial subgroup corresponds to the universal cover. Also if H and G are two subgroups of $\pi(X)$, such that $H \subset G$, then the cover corresponding to H acts as a covering space of the cover corresponding to G. A lot of similarity can already be seen between these two theories. A natural question to ask is, whether we can further talk about these two theories as completely one and try to solve problems in one by considering an analogous problem in the other.

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