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Additional Sheet

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<u>Free Product</u>: Let $\{G_\alpha : \alpha \in \Lambda\}$ be family of group. Then free product of $\{G_\alpha : \alpha \in \Lambda\}$ denoted by ~~Gα~~ $G = \underset{\alpha\in\Lambda}{*} G_\alpha$ is set of all finite reduced words over $\{G_\alpha\}_{\alpha\in\Lambda}$ with binary operation juxtaposition.

Reducible words means $x_1 x_2 \cdots x_n$, $x_i, x_{i+1} \notin G_\alpha$ $\forall\, 1 \le i \le n-1$

<u>Universal property</u>: Let $\psi_\alpha : G_\alpha \longrightarrow H$ be a family of group homomorphisms. ~~for~~ & Then there exist a unique group homomorphism $\psi : \underset{\alpha\in\Lambda}{*} G_\alpha \longrightarrow H$ such that for each $\alpha \in \Lambda$, the following diagram commutes

$$\begin{tikzcd}
G_\alpha \arrow[r, "\psi_\alpha"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & H \\
\underset{\alpha\in\Lambda}{*} G_\alpha \arrow[ur, "\psi"'] &
\end{tikzcd}$$

<u>Proof</u>:

Define $\psi(x_1 x_2 \cdots x_n) = \psi_{\alpha_1}(x_1)\, \psi_{\alpha_2}(x_2) \cdots \psi_{\alpha_n}(x_n)$

Consider the following diagram.

For a group A, G_1, G_2 and $\phi_1 : A \longrightarrow G_1$
$\phi_2 : A \longrightarrow G_2$

$$\begin{tikzcd} & G_1 \arrow[dr, hook] & \\ A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & & G_1 * G_2 \\ & G_2 \arrow[ur, hook] & \end{tikzcd}$$

Q: ~~When does this~~ Does this diagram commute?
or When does this diagram commute.

Ans: It commutes iff $\phi_1(a) = \phi_2(a) \quad \forall\, a \in A$.

So impositing ~~conditions~~ relations $\phi_1(a) = \phi_2(a)$ on $G_1 * G_2$

~~is equiva~~

i.e. ~~$\phi_1(a)\phi_2(a)^{-1}$ should be identity~~.

Thus let $N :=$ Normal subgroup generated by $\langle \phi_1(a)\phi_2(a)^{-1} \mid a \in A \rangle$

Then we have.

$$\begin{tikzcd} & G_1 \arrow[dr, hook] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[ur, "\phi_1"] \arrow[rr] \arrow[dr, "\phi_2"'] & {} \arrow[d, phantom, "\circlearrowleft"] & \dfrac{G_1 * G_2}{N} := G_1 *_A G_2 \\ & G_2 \arrow[ur, hook] & \end{tikzcd}$$

Universal Property for Amalgamated product

Let H be a group such that the following diagram commutes

$$\begin{tikzcd}
 & G_1 \arrow[dr, "l_1"] \arrow[d, phantom, "\circlearrowright"] & \\
A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] \arrow[rr, dotted, "f"] & {} \arrow[d, phantom, "\circlearrowleft"] & H \\
 & G_2 \arrow[ur, "l_2"'] &
\end{tikzcd}$$

Then the homomorphism from A, G_1, G_2 factors through $G_1 *_A G_2$

$$\begin{tikzcd}
G_1 \arrow[r, "l_1"] \arrow[d, hook] & H \\
G_1 * G_2 \arrow[r, "l"] & H \\
G_2 \arrow[u, hook] \arrow[ur, "l_2"'] &
\end{tikzcd}$$

s.t $\quad l(x_1x_2) = l_1(x_1)\, l_2(x_2)$.

For $a \in A$

$$l(\phi_1(a)\phi_2(a)^{-1}) = l_1\phi_1(a)\, l_2(\phi_2(a)^{-1})$$

$$= l_1(\phi_1(a))\, l_2(\phi_2(a))^{-1}$$

$$= e_H$$

$\therefore \quad N \subseteq \ker \cancel{\Phi}\, l$

$\therefore \quad \exists!\ \tilde{l} : G_1 *_A G_2 \longrightarrow H$

Thus we have the following commutative diagram.

$$\begin{tikzcd}
 & G_1 \arrow[d] \arrow[drr, "l_1"] & & \\
A \arrow[ur, "\phi_1"] \arrow[r, "\tilde{\phi}"] \arrow[dr, "\phi_2"'] & G_1 *_A G_2 \arrow[rr, "\tilde{l}"] & & H \\
 & G_2 \arrow[u] \arrow[urr, "l_2"'] & &
\end{tikzcd}$$

$$\tilde{l} \circ \tilde{\phi} = f.$$

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Main Theorem: Let X be a path connected topological space. Let $U, V, U \cap V$ be nonempty open path connected ~~subspaces~~ set of $X = U \cup V$. ~~Then~~ ~~or~~ Let $x_0 \in U \cap V$. Then

$$\pi_1(X, x_0) \simeq \pi_1(U, x_0) *_N \pi_2(V, x_0)$$

where $N = \pi_1(U \cap V, x_0)$.

Proof: Consider the following diagram

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr, "j_{U\#}"] \arrow[d, phantom, "\circlearrowright"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_{U\#}"] \arrow[rr] \arrow[dr, "i_{V\#}"'] & \, \arrow[d, phantom, "\circlearrowleft"] & \pi_1(X, x_0) \\
 & \pi_1(V, x_0) \arrow[ur, "j_{V\#}"'] &
\end{tikzcd}$$

$$j_{U\#} \circ i_{U\#}\left([\alpha]_{U \cap V}\right) = j_{U\#}\left([\alpha]_U\right) = [\alpha]_X$$

$$j_{V\#} \circ i_{V\#}\left([\alpha]_{U \cap V}\right) = j_{V\#}\left([\alpha]_V\right) = [\alpha]_X.$$

Thus by universal property of Amalgamated product we have a group homomorphism.

$$\phi : \pi_1(U, x_0) *_N \pi_2(V, x_0) \longrightarrow \pi_1(X, x_0)$$

We will show that ϕ is bijection.

$$[u_1]_U [v_1]_V \cdots [u_n]_U [v_n]_V \longmapsto u_1 * v_1 * \cdots * u_n * v_n$$

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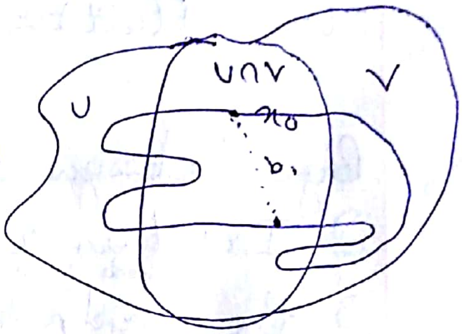
Onto: Let $[\alpha] \in \pi_1(X, x_0)$

$\alpha : [0,1] \longrightarrow X$.

By Lebesgue no. lemma, there exist a partition of $[0,1]$ such that $\alpha[t_i, t_{i+1}] \subseteq U$

or $\alpha[t_i, t_{i+1}] \subseteq V$



Obs: If $\alpha([t_i, t_{i+1}]) \subseteq U$

& $\alpha([t_{i+1}, t_{i+2}]) \subseteq V$

then $\alpha(t_{i+1}) \in U \cap V$.

Let b_j be path in $U \cap V$ from x_0 to $\alpha(t_j)$.

Let $\alpha|_{[t_i, t_{i+1}]} = \alpha_i$

Consider the word

$[\alpha_0 b_1^{-1}]_U \, [b_1 \alpha_1 b_2^{-1}]_V \cdots [b_{n-1} \alpha_{n-1} b_n]_U \, [b_n^{-1} \alpha_n]_V \in \pi_1(U, x_0) *_A \pi_1(V, x_0)$

$\phi [\alpha_0 b_1^{-1} * b_1 \alpha_1 b_2^{-1} \cdots * b_n b_n^{-1} \alpha_n]$

$= [\alpha_0 \alpha_1 \cdots \alpha_n]$

$= \alpha$.

Thus ϕ is onto.

One-one :

T.S.T $\ker\phi = \{e\}$ is trivial.

Let $[u_1]_U [v_1]_V \cdots [u_n]_U [v_n]_V \in \ker\phi$

i.e. $\phi(\ldots) = [c_{x_0}]_X$

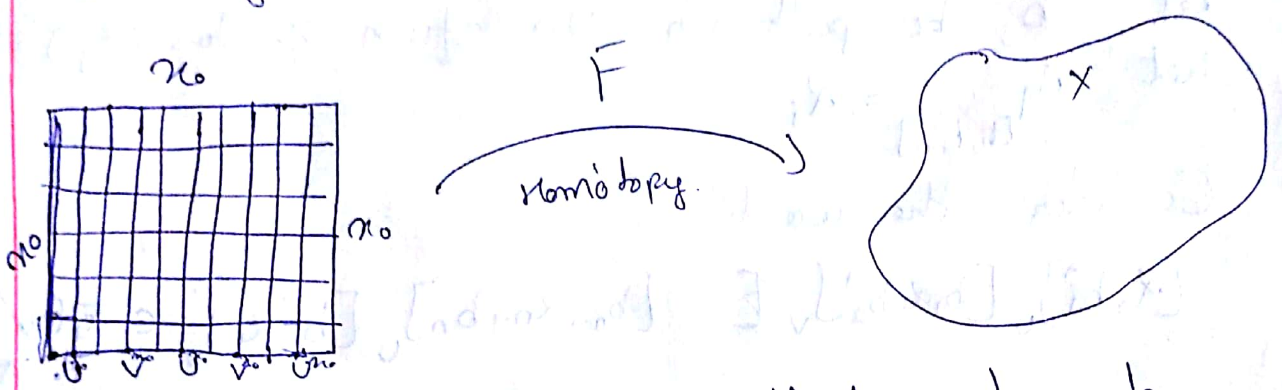
i.e. $\exists$ a homotopy $F : I \times I \longrightarrow X$

s.t ~~$F(0,t)$~~ $F(s,0) = u_1 * v_1 * \cdots * u_n * v_n$

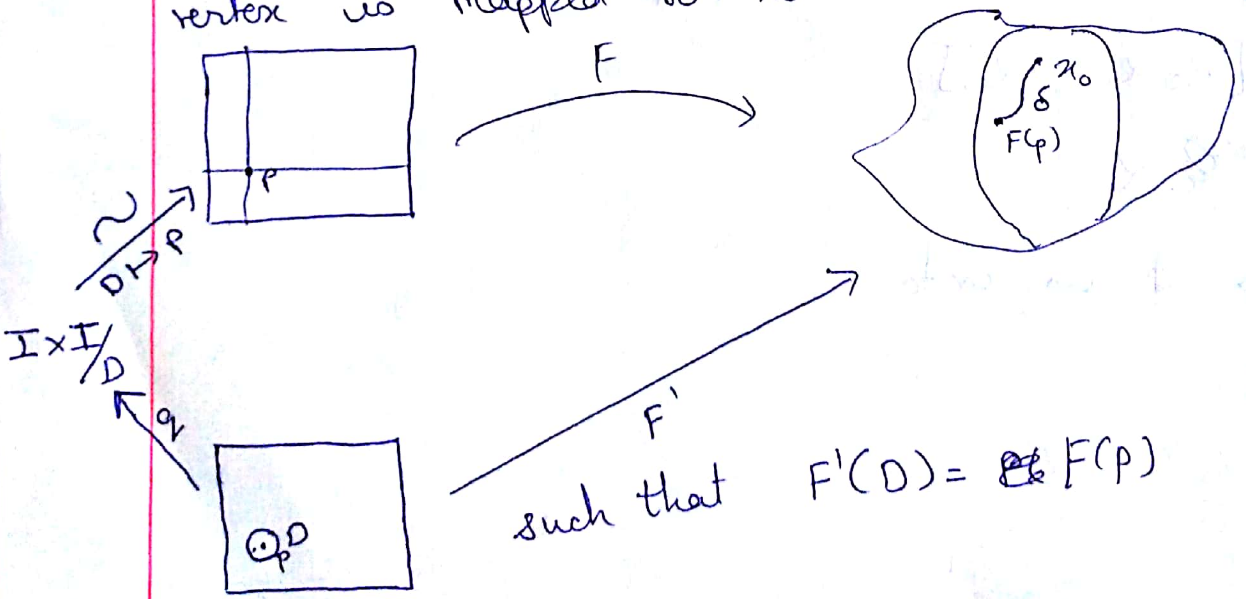
$F(s,1) = c_{x_0}$.

By Lebesgue lemma, there exist a partition of $I \times I$ in grids of side $\frac{1}{n}$ such that F maps ~~each square~~ of the grid either to U or V.

Refine the partition such that there are integral number of paths on interval of length $\frac{1}{n}$.



Modify the homotopy such that each each vertex is mapped to x_0.



such that $F'(D) = F(p)$

Now define

$$F'' : I \times I \longrightarrow X$$

as $F''(x) = F'(x) \quad \forall x \in (D^\circ)^c$

$$= \delta\left(\frac{x}{\varepsilon}\right) \quad \forall x \in D$$

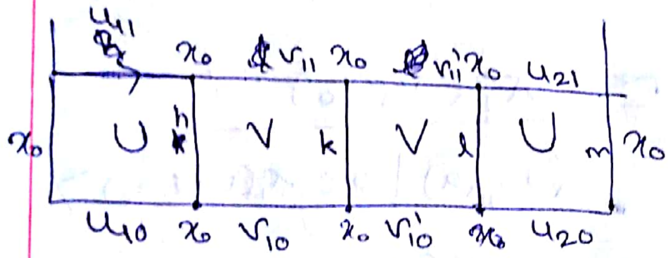
By Glueing lemma $F''(\partial D) = F(p)$

$$\delta(\partial D) = F(p)$$

F'' is continuous and $F''(p) = \delta(0) = x_0$

Carrying out the process at each step, each vertex of the squares of grid are mapped to x_0.

Consider the ~~level~~



$[u_{10}]_U = [u_{11} h]_U$, $[v_{10}]_V = [\bar{h} v_{11} k]$, $[v'_{10}]_V = [\bar{k} v'_{11} l]$, $u_{20} = [\bar{l} u_{21}]$

$$[u_1]_U [v_1]_V [v'_1]_V [u_2]_U = [u_{11} h]_U [\bar{h} v_{11} k]_V [\bar{k} v'_{11} l]_V [\bar{l} u_{21}]_U$$

$$= [u_{11} h]_U [\bar{h} v_{11} v'_{11} l]_V [\bar{l} u_{21}]_U$$

$$= [u_{11}]_U [h]_U [\bar{h}]_V [v_{11} v'_{11}]_V [l]_V [\bar{l}]_U [u_{21}]$$

$$= [u_{11}]_U [v_{11} v'_{11}]_V [u_{21}]_U$$

$$= [u_{11}]_U [v_{11}]_V [v'_{11}]_V [u_{21}]_U$$

Proceeding at each level, we show that the word at level 0 is equivalent to word at top level $[C x_0]$ which is trivial word in amalgamated product.

$\therefore$ ker ϕ is trivial.

Corollary 1: If $U \cap V$ is simply connected.

Then $\pi_1(U \cap V, x_0) = \{[c_{x_0}]\}$

$$\therefore \pi_1(X, x_0) \simeq \pi_1(U, x_0) * \pi_1(V, x_0)$$

Corollary 2: If one of the open set say ~~is~~ U is simply connected then

$\pi_1(U, x_0) = \{[c_{x_0}]\}$

$$\therefore \pi_1(U, x_0) *_N \pi_1(V, x_0) \simeq \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{N}$$

$$\simeq \frac{\pi_1(V, x_0)}{\langle i_{U\#}(a)\, i_{V\#}(a) \mid a \in \pi_1(U \cap V, x_0) \rangle}$$

$$\simeq \frac{\pi_1(V, x_0)}{\langle i_{V\#}(a) \mid a \in \pi_1(U \cap V, x_0) \rangle}$$

$$\simeq \frac{\pi_1(V, x_0)}{i_{V\#}(\pi_1(U \cap V, x_0))}$$

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