POLYA'S ENUMERATION

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ABSTRACT. We explore Polya's theory of counting from first principles, first building up the necessary algebra and group theory before proving Polya's Enumeration Theorem (PET), a fundamental result in enumerative combinatorics. We then discuss generalizations of PET, including the work of de Bruijn, and its broad applicability.

CONTENTS

1. INTRODUCTION

A common mathematical puzzle is finding the number of ways to arrange a necklace with n differently colored beads. Yet $(n-1)!$ and $\frac{(n-1)!}{2}$ are both valid answers, since the question has not defined what it means for necklaces to be distinct. The former counts the number of distinct necklaces up to rotation, while the latter counts the number of distinct necklaces up to rotation and reflection. Questions like these become more complex when we consider "distinctness" up to arbitrary transformations and with objects of more elements and non-standard symmetries. The search for a general answer leads us to concepts in group theory and symmetry, and ultimately towards Polya's enumeration, which we will explore below.

2. Basic Definitions and Properties

We start with one of the most basic algebraic structures:

Definition 2.1. Group. A group is a set G equipped with an operation $*$ satisfying the properties of associativity, identity, and inverse:

- Associativity: $\forall a, b, c \in G$, $(a * b) * c = a * (b * c)$.
- Identity: $\exists e \in G | \forall a \in G$, $e * a = a * e = a$.
- Inverse: $\forall a \in G$, $\exists a^{-1} \in G | a * a^{-1} = a^{-1} * a = e$.

Given group elements g, h in group G, we denote $g * h$ as gh.

Definition 2.2. Subgroup. A subgroup of a group G is a group under the same operation of G whose elements are all contained in G.

If H is a subgroup of G, we write $H \leq G$.

One of the most important groups is the **symmetric group** S_n, whose elements are all *permutations* of the set $\{1, ..., n\}$, and whose operation is composition. Indeed, permutations are associative under composition, there is an identity permutation, and all permutations have an inverse. Every finite set X also has an implied symmetric group $Sym(X)$, which simply involves all permutations of its elements.

Groups can also "act" on sets in the following manner:

Definition 2.3. Group action. Given a group G and a set X, a left group action is a function $\phi : G \times X \to X$ satisfying the properties of left identity and compatibility:

- Left Identity: For the identity element $e \in G$, for all $x \in X$, $\phi(e, x) = x$.
- Left Compatibility: For all $g, h \in G$, for all $x \in X$, $\phi(gh, x) = \phi(g, \phi(h, x))$.

A right group action is similarly defined as a function $\phi : X \times G \to X$ satisfying right identity and compatibility:

- Right Identity: For the identity element $e \in G$, for all $x \in X$, $\phi(x, e) = x$.
- Right Compatibility: For all $g, h \in G$, for all $x \in X$, $\phi(x, gh) = \phi(\phi(x, g), h)$.

Group actions will be left group actions unless specified otherwise, but the definitions and properties below apply analogously to right group actions as well. Given a group element g in a group G, an element x in a set X, and a group action ϕ, we denote $\phi(g, x)$ as gx if ϕ is a left group action and $\phi(x, g)$ as xg if ϕ is a right group action.

In any group action, the group also acts in a bijective manner on the set:

Proposition 2.4. *Given a group action ϕ of group G on a set X, the function $f_\phi : x \mapsto \phi(g, x)$ is bijective for all $g \in G$.*

Proof. It suffices to find an inverse function. We see that $h_\phi : x \mapsto g^{-1}x$ is such an inverse, since

$$f_\phi(h_\phi(x)) = f_\phi(g^{-1}x) = g(g^{-1}x) = (gg^{-1})x = x,$$
$$h_\phi(f_\phi(x)) = h_\phi(gx) = g^{-1}(gx) = (g^{-1}g)x = x$$

by compatibility of the group action. □

Thus, one may alternatively view the group action as associating a permutation

$p_g \in Sym(X)$ with every $g \in G$, where gx for $g \in G$ and $x \in X$ is determined by $p_g(x)$, the image of x in p_g. Formally, a group action ϕ is a homomorphism from G to $Sym(X)$. If we actually consider $G = Sym(X)$ as our group acting on X, then G naturally acts on X; that is, for $p_g \in G$, $\phi(p_g, x) = p_g(x)$ is the *natural* group action associated with G and X.

Associated with the elements of the set in any group action are two important notions:

Definition 2.5. Orbit. Given a group action ϕ of a group G on a set X, the orbit of a set element $x \in X$ is

$$orb(x) = O_x = \{gx : g \in G\} = \{y \in X | \exists g \in G : y = gx\}.$$

Definition 2.6. Stabilizer. Given a group action ϕ of a group G on a set X, the stabilizer of a set element $x \in X$ is

$$stab(x) = S_{xx} = \{g \in G | gx = x\}.$$

Using the stabilizer notation, we can similarly define the transformer:

Definition 2.7. Transformer. Given a group action ϕ of a group G on a set X, the transformer of two set elements $x, y \in X$ is

$$trans(x, y) = S_{xy} = \{g \in G | gx = y\}.$$

Associated with a group action is the set of orbits, called the quotient:

Definition 2.8. Quotient. Given a group action ϕ of a group G on a set X, the quotient of ϕ is defined as

$$X/G = \{O_x : x \in X\}.$$

As it turns out, the orbits of a set partition it:

Proposition 2.9. *For any group action ϕ of a group G on a set X, X/G is a partition of X.*

Proof. It is well-known that equivalence classes of a set partition it. Then it suffices to show that the relation $x{\sim}y \iff x, y \in O_x$ is an equivalence relation. We check the reflexive, symmetric, and transitive properties:

- Reflexive: For all $x \in X$, $x{\sim}x$ since $ex = x \in O_x$ for the identity element $e \in G$.
- Symmetric: For all $x, y \in X$, $x{\sim}y$ clearly implies $y{\sim}x$.
- Transitive: For all $x, y, z \in X$, if $x{\sim}y$ and $y{\sim}z$, then $x, y, z \in O_x$, so $x{\sim}z$ as well. □

It is also worth noting that the stabilizer of any element $x \in X$ forms a subgroup of G:

Proposition 2.10. *For any group action ϕ of a group G on a set X, $S_{xx} \leq G$ for all $x \in X$.*

Proof. Associativity is inherited from the group structure of G. We check the closure, identity, and inverse properties. For $g_i, g_j \in S_{xx}$ and $x \in X$:

- Closed: Clearly $g_i(g_j x) = g_i x = x$. But by the compatibility property of ϕ, we must also have $(g_i g_j)x = x \Rightarrow g_i g_j \in S_{xx}$.

- Identity: The identity $e \in G$ is in S_{xx} since $ex = x$.
- Inverse: Consider arbitrary $g_i \in S_{xx}$. Since $g_i x = x$, we also have $g_i^{-1}(g_i x) = g_i^{-1} x \Rightarrow g_i^{-1} x = (g_i^{-1} g_i)x = ex = x$ by compatibility of ϕ, so $g_i^{-1} \in S_{xx}$. □

3. Supporting Theorems

3.1. **Orbit-Stabilizer Theorem.** With our notions of orbits and stabilizers in hand, we prove the fundamental orbit-stabilizer theorem:

Theorem 3.1. Orbit – Stabilizer Theorem. *Given any group action ϕ of a group G on a set X, for all $x \in X$,*

$$|G| = |S_{xx}||O_x|.$$

Proof. Let $g \in G$ and $x \in X$ be arbitrary. We first prove the following lemma:

Lemma 1. For all $y \in O_x$, $|S_{xx}| = |S_{xy}|$.

Proof. It suffices to show a bijection between S_{xx} and S_{xy}. Let $g_{xx} \in S_{xx}$ and $g_{xy} \in S_{xy}$. Clearly $g_{xy}g_{xx}x = g_{xy}x = y$, so $g_{xy}g_{xx} \in S_{xy}$. In addition, by definition of S_{xy}, we have $g_{xy}x = y$, so multiplying by g_{xy}^{-1} gives us $g_{xy}^{-1}g_{xy}x = g_{xy}^{-1}y \implies ex = x = g_{xy}^{-1}y$ by compatibility; thus, $g_{xy}^{-1} \in S_{yx}$ and so $g_{xy}^{-1}g_{xy} \in S_{xx}$.

Consider any $h \in S_{xy}$. Since $g_{xy}g_{xx} \in S_{xy}$, we may define $\chi : S_{xx} \to S_{xy} : g_{xx} \mapsto hg_{xx}$. Since $g_{xy}^{-1}g_{xy} \in S_{xx}$, we may define $\psi : S_{xy} \to S_{xx} : g_{xy} \mapsto h^{-1}g_{xy}$, which is also an inverse for χ:

$$\chi(\psi(g_{xy})) = \chi(h^{-1}g_{xy}) = hh^{-1}s_{xy} = g_{xy}$$
$$\psi(\chi(g_{xx})) = \psi(hg_{xx}) = h^{-1}hg_{xx} = g_{xx}.$$

Thus χ is a bijection and $|S_{xx}| = |S_{xy}|$. □

By Lemma 1, we have $|S_{xy}| = |S_{xx}|$ for all $y \in O_x$. Now note that the sets $S_{xy} : y \in O_x$ must partition G; this follows from the definition of the orbit and the fact that the group action is a function.[1] Thus $|G| = |S_{xx}||O_x|$, as desired. ■

3.2. **Burnside's Lemma.** We can now calculate the order of the group, the size of the stabilizer of an arbitrary set element, or the size of that element's orbit given the other two quantities. However, one quantity of interest, the *number* of orbits, is still in complete question! The following theorem, now attributed to Cauchy in 1845, determines the number of orbits in terms of the order of the group and the number of fixed elements under the group action:

Theorem 3.2. Burnside′s Lemma. *Given a finite group G, a finite set X, and a group action ϕ of G on X, the number of distinct orbits is*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where $X^g = \{x \in X | gx = x\}$, the set of elements of X fixed by action by g.

[1]It is important to note the difference between this statement and Proposition 2.9. The proposition states that the different orbits partition the set, whereas here we state that given any one of those orbits, every group element acting on x gives exactly one element in O_x, and that all elements in O_x are covered.

Proof. We first note that

$$\sum_{g\in G}|X^g| = |(g,x)\in(G,X) : gx = x| = \sum_{x\in X}|S_{xx}|,$$

so we just need to show

$$|X/G| = \frac{1}{|G|}\sum_{x\in X}|S_{xx}|.$$

By the Orbit-Stabilizer Theorem, we have that $|S_{xx}| = \frac{|G|}{|O_x|}$, so

$$\frac{1}{|G|}\sum_{x\in X}|S_{xx}| = \frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|O_x|} = \sum_{x\in X}\frac{1}{|O_x|}.$$

Since orbits partition X by Proposition 2.9, we can split up X into disjoint orbits of X/G. Thus, we can rewrite our sum, where A is an orbit in X:

$$\sum_{x\in X}\frac{1}{|O_x|} = \sum_{A\in X/G}\sum_{x\in A}\frac{1}{|A|} = \sum_{A\in X/G}1 = |X/G|,$$

so $|X/G| = \frac{1}{|G|}\sum_{x\in X}|X^g|$, as desired. ■

4. Polya's Enumeration

4.1. **Prerequisites.** Polya's enumeration introduces functions f from a finite set X to a new finite set Y. Notation-wise, Y^X is the set of all functions $f : X \to Y$, represented as a set of ordered pairs $(x_i, f(x_i))$ for $x_i \in X$. For instance, if Y is a set of colors, then $f \in Y^X$ is a coloring of the elements in X, and Y^X/G is the number of distinct colorings of X under some group action of G on Y^X.

From this perspective, an action ϕ of G on X induces a natural group action ϕ' of G on Y^X, namely:

$$\phi' : (g,f) \mapsto f' = f\circ p_g^{-1} = \{(\phi(g,x), f(x))|x\in X\}$$

for $f \in Y^X$. Indeed, ϕ' satisfies identity and compatibility:

$$ef = \{(ex, f(x))|x\in X\} = \{(x, f(x))|x\in X\} = f,$$

$$g_1(g_2f) = g_1f' = g_1(\{(g_2x, f(x))|x\in X\}) = \{(g_1(g_2x), f'(g_2x))|x\in X\}$$

$$= \{(g_1g_2)x, f(x)|x\in X\} = (g_1g_2)f.$$

Throughout this section, we assume an implicit group action ϕ of a group G on a finite set X of size n, where ϕ is arbitrary. In addition, we assume Y is another finite set. To state Polya's Enumeration Theorem, we introduce some more machinery:

Definition 4.1. Type. Let p be a permutation on X. Then the type of p is the set $\{b_1, ..., b_n\}$, where b_i is the number of cycles of length i in the cycle decomposition of p.

Definition 4.2. Cycle index polynomial. The cycle index polynomial Z_ϕ of the group action ϕ is defined as [2]

$$Z_\phi(x_1, ..., x_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{n} x_i^{b_i(g)},$$

where $b_i(g)$ is the i^{th} element of the type of the implied permutation $p_g \in Sym(X)$.

Definition 4.3. Function equivalence. Two functions $f \in Y^X$ are said to be equivalent under the action of G ($f_1 \sim_G f_2$) if they are in the same orbit of ϕ', i.e. there exists $g \in G$ such that $f_2 = gf_1$.

By the proof of Proposition 2.9, function equivalence is an equivalence relation, so Y^X will have equivalence classes under function equivalence:

Definition 4.4. Configuration. A configuration is an equivalence class of the equivalence relation $\sim_G$ on Y^X.

Analogously, we have that every configuration c is just an orbit of ϕ', and that the set of configurations C is just Y^X/G under ϕ'.

To weight our functions differently, we can assign weights to elements in Y:

Definition 4.5. Weight. Let $w : Y \to \mathbb{R}$ be a weight assignment to each element in Y. [3] Then the weight of a function $f \in Y^X$ is defined as

$$W(f) = \prod_{x \in X} w(f(x)).$$

It follows that all functions in a configuration c have the same common weight, which we call the weight of the configuration $W(c)$:

Proposition 4.6. *All functions in a configuration have the same common weight.*

Proof. Consider arbitrary functions $f_1, f_2 \in Y^X$ in configuration C. Since $f_1 \sim_G f_2$, there exists $g \in G$ such that $f_1(gx) = f_2(x)$. In addition, from Proposition 2.4, we know every group element acting on a set permutes it, so $W(f) = \prod_{x \in X} w(f(x)) = \prod_{x \in X} w(f(gx))$ for any $g \in G$. Thus,

$$W(f_1) = \prod_{x \in X} w(f_1(x)) = \prod_{x \in X} w(f_1(gx)) = \prod_{x \in X} w(f_2(x)) = W(f_2). \square$$

Definition 4.7. Configuration Generating Function (CGF). Let C be the set of all configurations c. Then the CGF is defined as

$$F(C) = \sum_{c \in C} W(c).$$

[2]The standard notation is Z_G, but here we use Z_ϕ to explicitly show that the cycle index polynomial not only depends on the algebraic structure of the group G, but also its induced permutation on X through the group action ϕ.

[3]In general, we may replace $\mathbb{R}$ with any commutative ring.

4.2. **Theorem.** We are now equipped to tackle Polya's Enumeration Theorem (PET). We cover both the unweighted and weighted versions of the theorem; the first can be proved directly from Burnside's Lemma:

Theorem 4.8. Polya's Enumeration Theorem (Unweighted). *Let G be a group and X, Y be finite sets, where $|X| = n$. Then for any group action ϕ of G on X, the number of distinct configurations in Y^X is*

$$|C| = \frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)},$$

where $c(g)$ denotes the number of cycles in the cycle decomposition of $p_g \in Sym(X)$, the permutation of X associated with the action of g on X.

Proof. Since configurations are orbits of ϕ', we have $|C| = |Y^X/G|$ under ϕ'. We apply Burnside's Lemma to the finite set Y^X with group action ϕ', which states that

$$|Y^X/G| = \frac{1}{|G|} \sum_{g \in G} |(Y^X)^g|.$$

It remains to show that $|(Y^X)^g| = |Y|^{c(g)}$. But any function $f \in Y^X$ will remain constant under the action of g if and only if all elements in X in each cycle are assigned the same set element in Y. There are thus $|Y|$ choices of elements in Y for each of the $c(g)$ cycles in the cycle decomposition, and the result follows. ■

We now state the weighted version of PET:

Theorem 4.9. Polya's Enumeration Theorem (Weighted). *Let G be a group and X, Y be finite sets, where $|X| = n$. Let w be a weight function on Y. Then for any group action ϕ of G on X, the CGF is given by*

$$Z_\phi \left(\sum_{y \in Y} w(y), \sum_{y \in Y} w(y)^2, ..., \sum_{y \in Y} w(y)^n \right).$$

Proof. We first prove the following lemma:

Lemma 1. $|C| = \frac{1}{|G|} \sum_{g \in G} \left| \left\{ f \in Y^X | (\forall x \in X)(f(gx) = f(x)) \right\} \right|.$

Proof. Let ϕ'_R be the right group action on Y^X induced by ϕ:

$$\phi'_R : (f, g) \mapsto f'_R = f \circ p_g = \{(x, f(\phi(g, x))) | x \in X\},$$

where $f \in Y^X$ and $g \in G$. The result follows by applying Burnside's Lemma to Y^X under ϕ'_R, as in Theorem 4.8. □

We now take ϕ'_R to be our group action on Y^X. Let $A(\omega) = \{c \in C | W(c) = \omega\}$ be the set of all configurations with common weight ω. $S_{gg} = \left\{ f \in Y^X | f = fg \right\}$ is the set of all functions stabilized by g; let $S_{gg}(\omega) = \left\{ f \in Y^X | f = fg, W(f) = \omega \right\}$ be the set of all functions stabilized by g with common weight ω. Then by Lemma 1, we have

$$|A(\omega)| = \frac{1}{|G|} \sum_{g \in G} |S_{gg}(\omega)|.$$

We can also group our CGF by weights:

$$CGF = \sum_{c\in C} W(c) = \sum_{\omega} \omega |A(\omega)| = \frac{1}{|G|}\sum_{\omega}\sum_{g\in G} \omega |S_{gg}(\omega)|$$

by the above equality. Since our sum is finite, we can switch the order of summation:

$$CGF = \frac{1}{|G|}\sum_{g\in G}\sum_{\omega} \omega |S_{gg}(\omega)| = \frac{1}{|G|}\sum_{g\in G}\sum_{f\in S_{gg}} W(f).$$

G permutes X through the group action, so the corresponding permutation p_g for $g \in G$ has a cycle decomposition $C_1, ..., C_k$, where $k \leq n$. It follows that if $f \in S_{gg}$, then $f(x) = f(gx) = f(g^2x) = ...$ for all $x \in X, g \in G$ and f is constant on each cycle C_i in the cycle decomposition. Then we have

$$\sum_{f\in S_{gg}} W(f) = \sum_{f\in S_{gg}}\prod_{x\in X} w(f(x)) = \sum_{f\in S_{gg}}\prod_{i=1}^{k}\prod_{x\in C_i} w(f(x)) = \sum_{f\in S_{gg}}\prod_{i=1}^{k} w(f(x_i))^{|C_i|},$$

where $x_i \in C_i$. Let $|Y| = m$. Since we are summing over all $f \in S_{gg}$, we need to cover all possible assignments of $y \in Y$ to cycles C_i, so our expression becomes

$$\sum_{f\in S_{gg}} W(f) = \prod_{i=1}^{k}\Big(w(y_1)^{|C_i|} + ... + w(y_m)\Big)^{|C_i|} = \prod_{i=1}^{k}\sum_{y\in Y} w(y)^{|C_i|},$$

and plugging this into the CGF expression gives us

$$CGF = \frac{1}{|G|}\sum_{g\in G}\left(\prod_{i=1}^{k}\sum_{y\in Y} w(y)^{|C_i|}\right).$$

Regardless of cycle length, by definition of the type, there will be $b_j(g)$ cycles of length j, so our expression is

$$CGF = \frac{1}{|G|}\sum_{g\in G}\prod_{j=1}^{n}\left(\sum_{y\in Y} w(y)^j\right)^{b_j(g)} = Z_\phi\left(\sum_{y\in Y} w(y), \sum_{y\in Y} w(y)^2, ..., \sum_{y\in Y} w(y)^n\right). \blacksquare$$

Note that setting $w(y) = 1$ for all $y \in Y$ makes $W(f) = 1$ for all $f \in Y^X$ and gives us a CGF of $Z_G(|Y|, ..., |Y|)$, so the unweighted version of PET immediately follows.

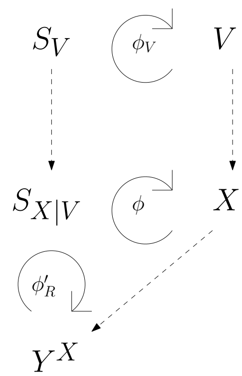
We summarize the concepts in PET with a concrete example:

Example 4.10. Classify the non-isomorphic multigraphs with $n = 4$ vertices and with up to $m = 2$ separate edges between two vertices allowed.

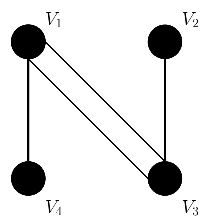
Solution. We first clarify our sets, groups and actions.

- Sets: Let V be the set of vertices $\{V_1, ..., V_n\}$, X be the set of edges $\{E_{12}, E_{13}, ..., E_{(n-1)(n)}\}$ of K_n, indicating all possible distinct edges, and Y be the set $\{y_0, ..., y_m\}$, indicating the number of possible edges between two vertices; let the weight of y_i be $w(y_i) = w_i$.
- Groups: We have $S_V = Sym(V)$ associated with V; let $S_{X|V}$ be the group of permutations on X induced by S_V. Note that this is not the same as $S_X = Sym(X)$, since $|S_{X|V}| = n!$ but $|S_X| = \binom{n}{2}!$.

- Actions: Let group S_V act on set V with the natural group action ϕ_V. Then group $S_{X|V}$ acts on set X through an induced action ϕ, and acts on set Y^X through an induced (right) action ϕ'_R as shown in the proof of PET.



Then Y^X represents all possible multigraphs, and $|Y^X/S_{X|V}|$ is the number of multigraphs up to isomorphism. For instance, the multigraph



is represented by the function $f : \{E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}\} \mapsto \{0, 2, 1, 1, 0, 0\}$.

We first compute the cycle index polynomial $Z_{\phi'_R}$. To do so, we need to determine the corresponding types to the elements of $S_{X|V}$. S_V acting on K_4 leads to the following elements in $S_{X|V}$:

- The identity $(V_1)(V_2)(V_3)(V_4) \in S_V$ leads to the corresponding identity $(E_{12})(E_{13})(E_{14})(E_{23})(E_{24})(E_{34}) \in S_{X|V}$ with type $\{6, 0, 0, 0, 0, 0\}$. This contributes an x_1^6 term to the cycle index.
- There are $\binom{4}{2} = 6$ elements of the form $(V_aV_b)(V_c)(V_d) \in S_V$ leading to the corresponding element $(E_{ab})(E_{cd})(E_{ac}E_{bc})(E_{ad}E_{bd}) \in S_{X|V}$ with type $\{2, 2, 0, 0, 0, 0\}$. Each of the 6 elements contributes an $x_1^2x_2^2$ term to the cycle index.
- There are $\frac{\binom{4}{2}}{2} = 3$ elements of the form $(V_aV_b)(V_cV_d) \in S_V$ leading to the corresponding element $(E_{ab})(E_{cd})(E_{ac}E_{bd})(E_{ad}E_{bc}) \in S_{X|V}$ with type $\{2, 2, 0, 0, 0, 0\}$. Each of the 3 elements contributes an $x_1^2x_2^2$ term to the cycle index.
- There are $\binom{4}{3} * 2 = 8$ elements of the form $(V_aV_bV_c)(V_d) \in S_V$ (note that $(123)(4)$ is different from $(132)(4)$) leading to the corresponding element $(E_{ab}E_{bc}E_{ac})(E_{ad}E_{bd}E_{cd}) \in S_{X|V}$ with type $\{0, 0, 2, 0, 0, 0\}$. Each of the 8 elements contributes an x_3^2 term to the cycle index.
- There are $3! = 6$ elements of the form $(V_aV_bV_cV_d) \in S_V$ leading to the corresponding element $(E_{ab}E_{bc}E_{cd}E_{ad})(E_{ac}E_{bd}) \in S_{X|V}$ with type $\{0, 1, 0, 1, 0, 0\}$. Each of the 6 elements contributes an x_2x_4 term to the cycle index.

Thus, the cycle index is

$$Z_{\phi'_R}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{24}(x_1^6 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2x_4).$$

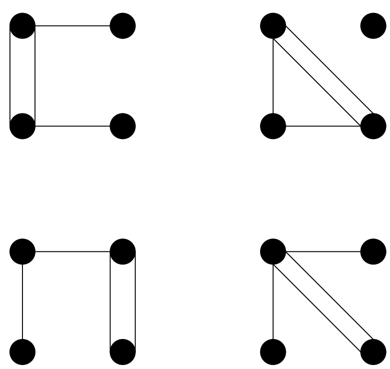
Now the weighted version of PET tells us that the CGF of Y^X/G is

$$Z_{\phi'_R}\left(\sum_{y\in Y} w(y), ..., \sum_{y\in Y} w(y)^6\right) = Z_{\phi'_R}\left((w_0 + w_1 + w_2), ..., (w_0^6 + w_1^6 + w_2^6)\right)$$

$$= \frac{1}{24}\left((w_0 + w_1 + w_2)^6 + 9(w_0 + w_1 + w_2)^2(w_0^2 + w_1^2 + w_2^2)^2 + 8(w_0^3 + w_1^3 + w_2^3)^2 + 6(w_0^2 + w_1^2 + w_2^2)(w_0^4 + w_1^4 + w_2^4)\right)$$

$$= w_0^6 + w_0^5w_1 + 2w_0^4w_1^2 + 3w_0^3w_1^3 + 2w_0^2w_1^4 + w_0w_1^5 + w_1^6 + w_0^5w_2 + 2w_0^4w_1w_2 + 4w_0^3w_1^2w_2 + 4w_0^2w_1^3w_2 + 2w_0w_1^4w_2 + w_1^5w_2 + 2w_0^4w_2^2 + 4w_0^3w_1w_2^2 + 6w_0^2w_1^2w_2^2 + 4w_0w_1^3w_2^2 + 2w_1^4w_2^2 + 3w_0^3w_2^3 + 4w_0^2w_1w_2^3 + 4w_0w_1^2w_2^3 + 3w_1^3w_2^3 + 2w_0^2w_2^4 + 2w_0w_1w_2^4 + 2w_1^2w_2^4 + w_0w_2^5 + w_1w_2^5 + w_2^6.$$

The CGF then completely classifies the non-isomorphic multigraphs of degree $n = 4$ with up to $m = 2$ separate edges between two vertices allowed. For instance, the term $4w_0^3w_1^2w_2$ indicates that there are four non-isomorphic multigraphs with 3 absent edges, 2 edges, and 1 double-edge, namely the multigraphs below:



If we set $w_0 = w_1 = 1, w_2 = 0$, we just get the number of non-isomorphic graphs of 4 vertices:

$$Z_{\phi'_R}(2, 2, 2, 2, 2, 2) = 11.$$

If we set $w_0 = w_1 = w_2 = 1$, we get the total number of non-isomorphic multigraphs:

$$Z_{\phi'_R}(3, 3, 3, 3, 3, 3) = 66.$$

If we set $w_0 = 0, w_1 = 1, w_2 = 2$, we get the total number of edges among all non-isomorphic multigraphs:

$$Z_{\phi'_R}(1 + 2, 1 + 2^2, ..., 1 + 2^6) = 163.$$

Other quantities of interest may be found by substituting different values for w_i. □

5. Extensions

Up until now, we have considered a group action ϕ with group G acting on set X, inducing an action ϕ' on the set Y^X. However, recall that $\phi' : (g, f) \mapsto f' = \{(\phi(g, x), f(x))\}$ permutes both X and $f(X)$ through ϕ. More generally, we can permute the set Y independently of ϕ; that is, we can consider an action ψ on set Y with another group H. We now define a more general equivalence between functions:

Definition 5.1. Generalized function equivalence. Two functions $f_1, f_2 \in Y^X$ are *equivalent* ($f_1 \sim_{gh} f_2$) if $\exists g \in G, h \in H$ such that for all $x \in X$, $f_1(gx) = hf_2(x)$.

For this definition to be compatible with our definitions of configuration, CGF, etc., $\sim_{gh}$ must be an equivalence relation. Proving this is left as an exercise to the reader.

Note that our previous proposition that equivalent functions have the same weight does not necessarily hold with generalized function equivalence; it is a requirement on ψ. However, assuming that generalized-equivalent functions have the same weight, we then have analogous results to the ones in section 4:

Theorem 5.2. Generalized PET (Weighted.) *Let groups G, H act on the sets X, Y through the group actions ϕ and ψ, respectively. Let $w : Y \to \mathbb{R}$ be a weight function for Y. Using generalized function equivalence, the CGF is*

$$CGF = \frac{1}{|G||H|} \sum_{(g,h)\in G\times H} \sum_{f\in S_{(g,h)}} W(f),$$

where $S_{(g,h)}$ is the set of functions $f \in Y^X$ stabilized by (g, h), i.e. $S_{(g,h)} = \left\{f \in Y^X |(\forall x \in X)(f(gx) = hf(x))\right\}$.

Proof. The proof follows exactly the same way as in the proof of PET (Weighted.) Note that function equivalence can also be written in the following form:

$$f_1 \sim_{gh} f_2 \iff (\exists (g, h) \in G \times H)(\forall x \in X)(h^{-1}f_1(gx) = f_2(x)).$$

Then we can define a right group action χ of group $G\times H$ on set Y^X in the following manner:

$$\chi(f, (g, h)) = h^{-1}fg,$$

where $g \in G$ acts on $f \in Y^X$ through the induced right group action $\phi'_R : (f, g) \mapsto f'_{\phi_R} = \{(x, f(\phi(g, x)))\}$ and $h \in H$ acts on $f \in Y^X$ through the induced left group action $\psi' : (h, f) \mapsto f'_\psi = \{(x, \psi(h, f(x)))\}$.

We then have that configurations are the orbits of $f \in Y^X$ under χ:

$$f_2 = \{(x, f_2(x))\} = f_1(g, h) = h^{-1}f_1g = h^{-1}(\{(x, f_1(gx))\}) = \left\{(x, h^{-1}f_1(gx))\right\}$$
$$\iff f_1 \sim_{gh} f_2.$$

Taking χ as our group action on Y^X, we again let

$$A(\omega) = \{c \in C|W(c) = \omega\},$$
$$S_{(g,h)}(\omega) = \left\{f \in Y^X|f(g, h) = f, W(f) = \omega\right\}.$$

By Burnside's Lemma with the group action χ, we have

$$|A(\omega)| = \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} |S_{(g,h)}(\omega)|.$$

Finally, recall that the CGF can be grouped into weights:

$$CGF = \sum_{c \in C} W(c) = \sum_{\omega} \omega |A(\omega)| = \frac{1}{|G \times H|} \sum_{\omega} \sum_{(g,h) \in G \times H} \omega |S_{(g,h)}(\omega)|$$
$$= \frac{1}{|G||H|} \sum_{(g,h) \in G \times H} \sum_{f \in S_{(g,h)}} W(f). \blacksquare$$

5.1. **De Bruijn's Theorem.** We finally arrive at the problem of counting the number of orbits, which involves weighting each function (up to generalized equivalence) with the weight 1. Before, we could simply substitute $w(y) = 1$ to get an answer of $Z_\phi(|Y|, ..., |Y|)$. Here, however, we no longer have a concise expression for the answer in terms of the CGF; we are looking for the quantity

$$\frac{\sum_{(g,h) \in G \times H} |f \in Y^X : f(g,h) = f|}{|G||H|},$$

which simply follows from Burnside's Lemma. We turn to de Bruijn's theorem:

Theorem 5.3. (**de Bruijn**.) *Let group G act on finite set X through group action ϕ, and let group H act on finite set Y through group action ψ. Then the number of functions up to function equivalence is*

$$Z_\phi\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...\right) Z_\psi\left(e^{\sum_k z_k}, e^{2\sum_k z_{2k}}, e^{3\sum_k z_{3k}}, ...\right)\Big|_{\{z_i\}=0}.$$

Proof. Let $b_i(g), c_j(h)$ be the types of $g \in G, h \in H$, respectively, and let $|X| = n, |Y| = m$. For ease of notation, let $b_i(g) = 0, c_j(h) = 0$ for all $g \in G, h \in H$ if $i > n$ and $j > m$, respectively, where i, j are taken over the positive integers $\mathbb{Z}^+$. We first prove the following lemmas:

Lemma 1. If $f \in S_{(g,h)}$, then $f(x) = y$ implies $f(g^i x) = h^i y$ for all i.

Proof. Since $fg = hf$, we have

$$fg^2 = (fg)g = (hf)g = h(fg) = h(hf) = h^2 f.$$

The result easily follows by induction on i:

$$fg^{i-1} = h^{i-1} f \Rightarrow fg^i = (fg^{i-1})g = h^{i-1}fg = h^{i-1}hf = h^i f. \square$$

Lemma 2. If $f \in S_{(g,h)}$, then each cycle C_X in X is mapped by f to a cycle C_Y in Y where $|C_Y|$ divides $|C_X|$.

Proof. Consider any $f \in S_{(g,h)}$, where $fg = hf$ for some $g \in G, h \in H$. Let $x \in X$ belong to cycle $C_{g,x}$ of length j. Then we have

$$C_{g,x} = \left\{x, gx, ..., g^{j-1}x\right\},$$

where $g^j x = x$. By Lemma 1, we have $f(g^i x) = h^i f(x)$ for all positive integers i. Thus, the image of $C_{g,x}$ under f is

$$f(C_{g,x}) = \left\{f(x), hf(x), h^2 f(x), ..., h^{j-1} f(x)\right\},$$

and we have $h^j f(x) = f(g^j x) = f(x)$, so the cycle C_Y in Y containing $f(x)$ must have a length that divides j. $\square$

Lemma 3. The total number of functions stabilized by (g,h) is

$$\sum_{f\in S_{(g,h)}} W(f)=\prod_i \left(\sum_{j|i} jc_j(h)\right)^{b_i(g)}.$$

(Recall that we are setting $W(f)=1$.)

Proof. We count the number of functions using the condition in Lemma 2. Let $f\in S_{(g,h)}$. For each cycle $C_{g,i}$ in the cycle decomposition of $p_g\in Sym(X)$, pick an arbitrary element $x_i\in C_{g,i}$. Since there are $c_j(h)$ cycles of length j in the cycle decomposition of $p_h\in Sym(Y)$, and $C_{g,x}$ can only map to cycles whose length divides its own by Lemma 2, x_i can map to $y\in Y$ under f in $\sum_{j|i} jc_j(h)$ ways. But note that after the mapping $x_i\mapsto f(x_i)$ has been determined, the rest of the mappings in $C_{g,i}$ are determined as well, due to the condition in Lemma 1. Thus, the number of functions is the number of ways to choose mappings for each cycle in p_g; since p_g has $b_i(g)$ cycles of length i, the result follows. □

By Lemma 3, we have

$$\sum_{f\in S_{(g,h)}} W(f)=(c_1(h))^{b_1(g)}\cdot(c_1(h)+2c_2(h))^{b_2(g)}\cdot(c_1(h)+3c_3(h))^{b_3(g)}...$$

But note that each of the terms of the form a^b in this product can be written as the partial derivative expression $\frac{\partial^b}{\partial z^b}e^{az}\Big|_{z=0}$:

$$\left(\sum_{j|i} jc_j(h)\right)^{b_i(g)}=\frac{\partial^{b_i(g)}}{\partial z_i^{b_i(g)}}e^{(\sum_{j|i} jc_j(h))z_i}\Big|_{z_i=0}.$$

More generally, $a^b=\frac{\partial^b}{\partial z^b}e^{a(\sum_i z_i)}\Big|_{\{z_i\}=0}$, so we can write our expression as

$$\sum_{f\in S_{(g,h)}} W(f)=\left(\prod_i(\frac{\partial}{\partial z_i})^{b_i(g)}\right)\ e^{(\sum_{j|i} jc_j(h))(\sum_i z_i)}\Big|_{\{z_i\}=0}.$$

We can also express the exponent solely in terms of j:

$$\left(\sum_{j|i} jc_j(h)\right)\left(\sum_i z_i\right)=\left(\sum_j jc_j(h)\sum_{k=1}^{\infty} z_{kj}\right)$$

Let $r_i=\frac{\partial}{\partial z_i}$, $s_j=e^{j\sum_k z_{kj}}$. Then we have

$$\frac{1}{|G|}\sum_{g\in G}\prod_i(\frac{\partial}{\partial z_i})^{b_i(g)}=Z_\phi(r_1,r_2,...),$$

$$\frac{1}{|H|}\sum_{h\in H} e^{(\sum_{j|i} jc_j(h))(\sum_i z_i)}=\frac{1}{|H|}\sum_{h\in H} e^{\sum_j jc_j(h)\sum_{k=1} z_{kj}}$$

$$=\frac{1}{|H|}\sum_{h\in H}\prod_j(e^{jc_j(h)\sum_k z_{kj}})=\frac{1}{|H|}\sum_{h\in H}\prod_j(e^{j\sum_k z_{kj}})^{c_j(h)}=Z_\psi(s_1,s_2,...).$$

Finally, by Generalized PET, we have

$$CGF = \frac{1}{|G||H|}\sum_{(g,h)\in G\times H}\sum_{f\in S_{(g,h)}} W(f)$$

$$= \left(\frac{1}{|G|}\sum_{g\in G}\prod_i (\frac{\partial}{\partial z_i})^{b_i(g)}\right)\left(\frac{1}{|H|}\sum_{h\in H} e^{(\sum_{j|i} jc_j(h))(\sum_i z_i)}\right)$$

$$= Z_\phi(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...)Z_\psi(e^{\sum_k z_k}, e^{2\sum_k z_{2k}}, e^{3\sum_k z_{3k}}, ...)\Big|_{\{z_i\}=0}. \blacksquare$$

Here is a simple example that highlights the differences between PET and its generalizations:

Example 5.4. Compute the number of distinct colorings of the vertices of a square with 3 colors, under the following equivalencies:

- 1. Rotations are not distinct.
- 2. Rotations and reflections are not distinct.
- 3. Rotations, reflections, and color permutations are not distinct.

Solution. The first two cases can be solved using Burnside's Lemma and PET; the third case involves generalized PET and de Bruijn's Theorem. Our sets are $X = \{V_1, V_2, V_3, V_4\}$, the set of vertices, and $Y = \{R, G, B\}$, our three colors. Let $r = (V_1V_2V_3V_4)$ be a clockwise 90-degree rotation and $s = (V_1V_2)(V_3V_4)$ be a reflection across the vertical axis. The corresponding groups for each case are

- 1. $G = C_4 = \{e, r, r^2, r^3\}$, the cyclic group on the vertices,
- 2. $G = D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, the dihedral group on the vertices,
- 3. $G = D_4$ and $H = S_3$, the symmetric group on the colors,

where the group actions are the natural group actions.

For the sake of demonstration, we solve case 1 and 2 in two different ways. By Burnside's Lemma, the answer to the first case is

$$|Y^X/G| = \frac{1}{|G|}\sum_{g\in G}(Y^X)^g = \frac{1}{4}(3^4 + 3^1 + 3^2 + 3^1) = 24.$$

For the second case, we compute the cycle index for the natural group action $\phi_2 : D_4 \times X \to X$:

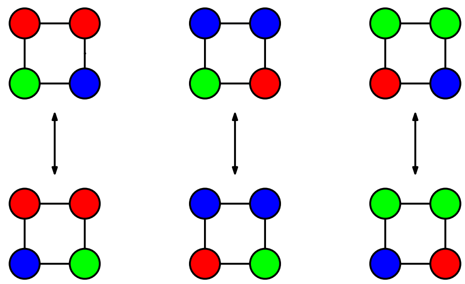
$$Z_{\phi_2}(x_1, x_2, x_3, x_4) = \frac{1}{|D_4|}\sum_{g\in D_4}\prod_{i=1}^{4} x_i^{(b_i(g))} = \frac{1}{8}(x_1^4 + x_4 + x_2^2 + x_4 + x_2^2 + x_1^2x_2 + x_2^2 + x_1^2x_2)$$

$$= \frac{1}{8}(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4).$$

The number of distinct colorings is then equal to

$$Z_{\phi_2}(3, 3, 3, 3) = \frac{1}{8}(3^4 + 2(3^2)(3) + 3(3^2) + 2(3)) = 21.$$

The three coloring-pairs distinct under rotation but not rotation and reflection are shown below:



Finally, for the last case, we use de Bruijn's theorem. We have

$$Z_\phi(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4)$$

$$Z_\psi(x_1, x_2, x_3) = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3),$$

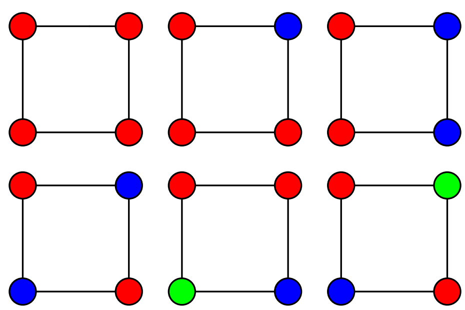
so the number of distinct colorings is equal to

$$Z_\phi(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...)Z_\psi(e^{\sum_{i=1} z_i}, e^{2\sum_{i=1} z_{2i}}, e^{3\sum_{i=1} z_{3i}}, ...)\Big|_{\{z_i\}=0}$$

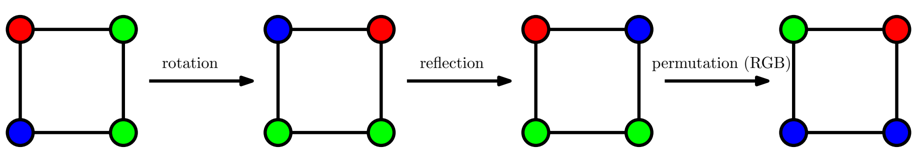
$$= \frac{1}{24}(\frac{\partial^4}{\partial z_1^4} + \frac{\partial^2}{\partial z_2^2} + 2\frac{\partial}{\partial z_4})(e^{3(z_1+z_2+z_3+z_4)} + 3e^{z_1+z_2+z_3+z_4}e^{2z_2+2z_4} + 2e^{3z_3})\Big|_{\{z_i\}=0}$$

$$= \frac{1}{24}(3^4 + (3) + 0 + 3^2 + (3)(3^2) + 0 + (2)(3) + (2)(3)(3) + 0) = 6. \square$$

The 6 distinct colorings are shown below:



where all of the colorings below are equivalent to one another:



6. Further Work

Multiple generalizations of Polya's Enumeration Theorem exist, most coming from the work of de Bruijn, that are not fully addressed in this paper. As more generalized theorems of PET are developed, the most meaningful work on the subject may certainly come from clever substitutions or cases of these theorems. More modern applications of the theorem can be found in the fields of analytic combinatorics and random permutation statistics, among others.

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